Minimal Ahlfors regular conformal dimension of coarse conformal dynamics on the sphere

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Abstract

We prove that if the Ahlfors regular conformal dimension Q of a topologically cxc map on the sphere $f : S^2 \to S^2$ is realized by some metric d on S^2, then either $Q = 2$ and f is topologically conjugate to a semihyperbolic rational map with Julia set equal to the whole Riemann sphere, or $Q > 2$ and f is topologically conjugate to a map which lifts to an affine expanding map of a torus whose differential has distinct real eigenvalues. This is an analog of a known result for Gromov hyperbolic groups with two-sphere boundary, and our methods apply to give a new proof.

Contents

1 Introduction

A topologically coarse expanding conformal (cxc) orientation-preserving branched covering map $f : S^2 \to S^2$ is an analog, in the setting of iterated maps, of a Gromov-hyperbolic group G whose boundary is the two-sphere. Let $\mathcal{D}$ denote the dynamical system on S^2 given either by the iterates of f or by the elements of G. In both settings, there is a canonical quasisymmetry class $\mathcal{G}(\mathcal{D})$, or conformal gauge, of Ahlfors-regular metrics on S^2 in which the elements of $\mathcal{D}$ have uniformly bounded metric distortion. With respect to a metric $d \in \mathcal{G}(\mathcal{D})$, for maps, one says that f is metrically cxc, while for groups, one says that the action of G is uniformly quasi-Möbius.

When is $\mathcal{D}$ topologically conjugate to a genuine conformal dynamical system, i.e. to a rational map or to a Kleinian group? The topological assumptions imply that such a rational map is necessarily semihyperbolic and that such a Kleinian group acts cocompactly when extended to act on hyperbolic three-space. For groups, Cannon's Conjecture asserts that the answer is always yes [Can]. For maps, this is no longer the case: there are combinatorial obstructions to Euclidean conformality, discovered by Thurston [DH].

The Ahlfors regular conformal dimension $\mathrm{confdim}_{AR}(\mathcal{D})$ of $\mathcal{D}$ is defined as the infimum of the set of Hausdorff dimensions $\mathrm{H.dim}(S^2, d)$ of metrics $d \in \mathcal{G}(\mathcal{D})$. The general concept was introduced by Pansu, who computed a closely related invariant for the natural metrics on the boundary at infinity of certain homogeneous manifolds of negative curvature [Pan]. Since the topological dimension is always a lower bound, in our setting one has $\mathrm{confdim}_{AR}(\mathcal{D}) \geq 2$. If $\mathcal{D}$ is given by a group, then conjecturally, $\mathrm{confdim}_{AR}(\mathcal{D}) = 2$. If $\mathcal{D}$ is given by a map, however, then the combinatorics of obstructions provide lower bounds on $\mathrm{confdim}_{AR}(\mathcal{D})$ that may be strictly larger than 2 [HP2].

A priori, the infimum in the definition may, or may not, be realized. For groups with boundary S^2, Bonk and Kleiner [BnK3, Thm. 1.1] have proven that if the infimum is achieved, then the conclusion of Cannon's Conjecture holds. For a general hyperbolic group, if the infimum is realized, then the group again has special properties; see [Kle] for a survey of results. The main result of this work is an analogous statement in the setting of maps: it gives a topological characterization of when the infimum is realized.

Theorem 1.1 (Rational or Lattès) *If $f : S^2 \to S^2$ is topologically cxc and a metric $d \in \mathcal{G}(f)$ realizes $\mathrm{confdim}_{AR}(f)$, then f is topologically conjugate to either*

1. *a semihyperbolic rational map, in which case* $\mathrm{confdim}_{AR}(f) = 2$ *, or*
2. *an obstructed Lattès example induced by an affine map on the torus whose differential has distinct positive real eigenvalues each larger than one, in which case* $\mathrm{confdim}_{AR}(f) > 2$.

The Lattès examples lift to covering maps of tori. They are a ubiquitous family of exceptional cases to general statements in the dynamics of rational maps. Our theorem is additional evidence that they play a similar role for the dynamics of cxc maps; cf. [MtM] and also the recent preprint [Yin]. Their conformal gauges are related to those arising from visual boundaries of certain three-dimensional solvable Lie groups.

We will derive Theorem 1.1 from the following general result, which applies to both maps and groups:

Theorem 1.2 *Suppose $\mathcal{D}$ is the dynamical system on S^2 determined by a topologically cxc map or a Gromov hyperbolic group. If* $\mathrm{confdim}_{AR}(\mathcal{D})$ *is attained, then either*

1. *$\mathcal{D}$ is topologically conjugate to a semihyperbolic rational map or cocompact Kleinian group, or*

2. *$\mathcal{D}$ preserves a foliation of S^2 having finitely many singularities.*

The two cases are not mutually exclusive; the overlap consists of so-called integral, or flexible, Lattès examples: the corresponding torus maps are conjugate to group endomorphisms of the form $x \mapsto m \cdot x$ where m is an integer with $|m| > 1$.

Theorem 1.2 yields an alternative proof of the result of Bonk and Kleiner mentioned above:

Theorem 1.3 (M. Bonk & B. Kleiner) *Let G be a hyperbolic group with boundary homeomorphic to S^2. Assume that its Ahlfors-regular conformal dimension is attained. Then the action of G on its boundary is topologically the action of a cocompact Kleinian group.*

The proof of Theorem 1.2 relies on a new characterization of the gauge of the Euclidean two-sphere $\mathbb{S}^2$ among those gauges on S^2 supporting such dynamical systems $\mathcal{D}$: this gauge is the only such gauge containing a metric in which two rectifiable thick curves cross; see the Dichotomy in § 1.4 below, § 4.2, and Proposition 4.17.

In the remaining subsections of this introduction, we give precise definitions and outline the proofs.

1.1 Topologically cxc maps

Throughout this work, f denotes a continuous, orientation-preserving, branched covering map from the sphere to itself with degree $\deg(f) \geq 2$. The following class of dynamical systems was introduced in [HP1].

Definition 1.4 (Topologically cxc) *The map f is called* topologically cxc *provided there exists a finite open covering $\mathcal{U}_0$ of S^2 by connected sets satisfying the following properties:*

> **[Exp]** *The mesh of the coverings $\mathcal{U}_n$ tends to zero as $n \to \infty$, where $\mathcal{U}_n$ denotes the set of connected components of $f^{-n}(U)$ as U ranges over $\mathcal{U}_0$.*

[Irred] *The map f is* locally eventually onto*: for any $x \in S^2$, and any neighborhood W of x, there is some n with $f^n(W) = S^2$.*

[Deg] *The set of degrees of maps of the form $f^k|\widetilde{U} : \widetilde{U} \to U$, where $U \in \mathcal{U}_n$, $\widetilde{U} \in \mathcal{U}_{n+k}$, and n and k are arbitrary, has a finite maximum $p < \infty$.*

We denote by $\mathbf{U} = \cup_{n\geq 0}\mathcal{U}_n$.

Note that the definition prohibits recurrent or periodic branch points. Also, one may choose $\mathcal{U}_0$ so that each atom of each cover $\mathcal{U}_n$ is a Jordan domain.

A rational map is topologically cxc on the sphere if and only if it is semihyperbolic with Julia set the whole sphere, i.e. has neither nonrepelling cyles nor recurrent critical points [HP1, Cor. 4.4.2].

The *postcritical set* of f is $P_f = \cup_{n>0} f^n(B_f)$, where B_f is finite the set of branch points of f. If P_f is finite, the *orbifold* associated to f has weight function $\nu : S^2 \to \mathbb{N}$ given by $\nu(y) = \mathrm{lcm}\{\deg(f^n, x) : f^n(x) = y, n \geq 1\}$ where $\deg(,)$ is the local degree; cf. [DH].

1.2 Conformal gauges

A homeomorphism $h : X \to Y$ between metric spaces is *quasisymmetric* provided there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that $d_X(x, a) \leq t \cdot d_X(x, b) \Rightarrow d_Y(f(x), f(a)) \leq \eta(t) \cdot d_Y(f(x), f(b))$ for all triples of points $x, a, b \in X$ and all $t \geq 0$.

The conformal gauges associated to maps and to groups possess many metric and dynamical regularity properties.

A metric space X is *Ahlfors regular* of dimension Q provided there is a Radon measure μ such that for any $x \in X$ and $r \in (0, \mathrm{diam}X]$,

$$\mu(B(x, r)) \asymp r^Q\,.$$

If this is the case, this estimate also holds for the Q-dimensional Hausdorff measure.

The next concept is due to Mackay [Mac2]. Suppose $L \geq 1$. A metric space X is *L-annularly linearly connected (ALC)* provided

(BT) for any x and y in X, there is a continuum K containing both points such that $\mathrm{diam}K \leq L|x - y|$;

(ALC2) for any $x \in X$, and $0 < r \leq 2r \leq R \leq \mathrm{diam}X$, any pair of points in $B(x, R) \setminus B(x, r)$ can be joined by a continuum K contained in $B(x, LR) \setminus B(x, r/L)$.

The initials (BT) stand for *bounded turning*. Such a metric space has no local cut points, and the continuum in condition (BT) can be taken to be an arc, indeed, a quasi-arc [Mac1]. The portion of the next result dealing with maps summarizes [HP1], Proposition 3.3.2 and Theorems 3.5.3 and 3.5.6; the portion dealing with groups may be found in [Pau] and [Mac2].

Theorem 1.5 (Canonical Gauge) *Given a dynamical system $\mathcal{D}$ as above, we may endow S^2 with a distance d_v and a measure μ with the following properties:*

1. *the space (S^2, d_v, μ) is Ahlfors regular, annularly linearly locally connected and doubling;*

2. *the measure μ is quasi-invariant by $\mathcal{D}$, i.e. sets of measure zero are preserved;*

3. *$(\mathcal{D} = G)$ the action of G is uniformly quasi-Möbius: there exists an increasing homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all distincts quadruples x_1, x_2, x_3, x_4 and all $g \in G$,*

$$[g(x_1) : g(x_2) : g(x_3) : g(x_4)] \le \eta([x_1 : x_2 : x_3 : x_4])$$

where

$$[x_1 : x_2 : x_3 : x_4] = \frac{d_v(x_1, x_2) d_v(x_3, x_4)}{d_v(x_1, x_3) d_v(x_2, x_4)}\,;$$

$(\mathcal{D} = \{f^n\})$ there exist constants $\theta \in (0, 1)$ and $r_0 > 0$ with the following properties: $\mathrm{diam}_v U \asymp \theta^n$ if $U \in \mathcal{U}_n$, $f^k(B_v(x, r\theta^k)) = B_v(f^k(x), r)$ for any $r < r_0$.

Furthermore, if d is another metric sharing these properties, then the identity map between (S^2, d_v) and (S^2, d) is quasisymmetric.

It follows that the set $\mathcal{G}(\mathcal{D})$ of all Ahlfors regular metric spaces Y quasisymmetrically equivalent to (S^2, d_v) is an invariant, called the *Ahlfors regular conformal gauge*, of the topological conjugacy class of $\mathcal{D}$; see [HP1, Thm. 3.5.3, Thm. 3.5.6]. Therefore, the *Ahlfors regular conformal dimension*

$$\mathrm{confdim}_{AR}(\mathcal{D}) := \inf_{Y \in \mathcal{G}(\mathcal{D})} \mathrm{H.dim}(Y)$$

is a numerical topological dynamical invariant as well. Moreover, this invariant almost characterizes conformal dynamics among topological ones on the sphere; see [HP1, Thm. 4.2.11] and [BnK1, Thm. 1.1], [BnK2, Thm. 1.1].

Theorem 1.6 (Characterization of conformal dynamical systems) *The dynamical system $\mathcal{D}$ is topologically conjugate to a semihyperbolic rational map or cocompact Kleinian group if and only if $\mathrm{confdim}_{AR}(f) = 2$ and is realized.*

This is a consequence of Bonk and Kleiner's characterisation of the Riemann sphere (Theorem A.1 in the Appendix below) and of Sullivan's straightening of uniformly quasiconformal groups and quasiregular maps.

1.3 Real Lattès maps

Let Γ be the subgroup of isometries of the plane generated by $x \mapsto x+(1,0), x \mapsto x+(0,1)$, and $x \mapsto -x$. Let $\mathbb{Z}^2 \subset \mathbb{R}^2$ be the standard integer lattice, and let A be a 2-by-2 integer matrix.

The quotient space $\mathbb{R}^2/\Gamma$ is homeomorphic to S^2. The map $\tilde{f}_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $v \mapsto Av$ descends to a branched covering $f_A : S^2 \to S^2$. The collection of such maps f_A generalizes the well-studied family of rational functions discovered by Lattès, so we call these *Lattès* maps. If the eigenvalues of A are real, we say that f_A is a *real Lattès map.*

The following facts are easily verified. A Lattès map f_A is topologically cxc if and only if A is expanding, i.e. all eigenvalues of A lie outside the closed unit disk. The postcritical set P_{f_A} of f_A has at most four points, with equality if and only if the orbifold of f_A has weight 2 at each point of P_{f_A}. Conversely, any critically finite topologically cxc map $f : S^2 \to S^2$ whose orbifold has signature $(2,2,2,2)$ is topologically conjugate to some Lattès map f_A: the map f lifts to an expanding map on the canonical double covering torus, and such toral maps are classified up to topological conjugacy by their action on homology. Note that the linear map given by A is determined only up to sign, cf. [DH, Prop. 9.3].

We will show

Theorem 1.7 *Let f_A be a Lattès map with associated matrix $\pm A$.*

1. *If A is a multiple of the identity or has non-real eigenvalues, then f_A is topologically conjugate to a rational function, and* $\mathrm{confdim}_{AR}(f) = 2$ *and is attained.*

2. *If A is semisimple with real eigenvalues $1 < |\lambda| < |\mu|$, then* $\mathrm{confdim}_{AR}(f_A) = 1 + \log|\mu|/\log|\lambda|$ *and is attained.*

3. *If A has a single repeated eigenvalue $|\lambda| > 1$, then* $\mathrm{confdim}_{AR}(f_A) = 2$ *and is not attained.*

The above classification of Lattès maps is intimately related to the classification of homogeneous three-manifolds of negative curvature.

Heintze [Hei] has classified all homogeneous manifolds of negative sectional curvature: they are solvable Lie groups obtained as an extension of a nilpotent group by the group $\mathbb{R}$ associated to a derivation α. In [Pan], Pansu computes the conformal dimension of its boundary at infinity when α is semi-simple, and gives a metric of minimal dimension. We complete his computations when α is not semi-simple in the three-dimensional case, cf. [BnK3, § 6]:

Theorem 1.8 *Let M be a homogeneous three-manifold of negative sectional curvature given by a non-semi-simple derivation on $\mathbb{R}^2$. Then the Ahlfors-regular conformal dimension of its boundary is 2, but is not attained.*

1.4 Outline of proofs

We begin by analyzing weak tangents T of metric spheres $X = (S^2, d) \in \mathcal{G}(\mathcal{D})$.

We first prove a general result: any weak tangent T of an arbitrary doubling and ALC metric surface is homeomorphic to the plane (Theorem 3.2). In the particular case of $X \in \mathcal{G}(\mathcal{D})$, the selfsimilarity of X induced by the dynamics implies that T is equipped with a locally quasisymmetric map $h : T \to X$; for groups, h is a homeomorphism to the complement in X of some point, while for maps, it is a surjective branched covering map.

Suppose the Ahlfors regular conformal dimension, Q, is attained by $X \in \mathcal{G}(\mathcal{D})$. A theorem of Keith and Laakso [KL] implies the existence of a weak tangent T_1 of X which contains a family of curves of positive Q-modulus. By a theorem of Tyson, the associated map $h_1 : T_1 \to X$ transports this family to such a family on X. Since Q-modulus behaves like an outer measure on the separable metric space of curves, there are "density points", called *thick curves*, on X. The collection of thick curves on X is invariant under the dynamics.

Two thick curves *cross* if any curve sufficiently close to one intersects any curve sufficiently close to the other. The key point in the proof of Theorem 1.2 is the following

Dichotomy.

1. If there are thick curves which cross, then $Q = 2$, X is quasisymmetrically equivalent to the Euclidean round sphere, and the dynamics is conjugate to the action of a discrete cocompact group of Möbius transformations or to a rational map.

2. If there are no thick curves which cross, then there is a foliation $\mathcal{F}$ of X by locally thick curves such that $\mathcal{F}$ has finitely many singularities and is invariant under the dynamics.

We prove (1) by means of a new estimate relating combinatorial moduli in different dimensions (Proposition 4.10 and Corollary 4.16) and a comparison theorem relating combinatorial and analytic moduli (Proposition 4.14). To prove (2), we show that a tangent T_2 of X at a suitable density point on a thick curve is foliated by curves whose images under the map $h_2 : T_2 \to X$ are locally thick and which, by assumption, cannot cross; this yields the foliation $\mathcal{F}$ on X.

For groups, case (2) cannot occur since the action of G is minimal on X. For maps, the invariance of $\mathcal{F}$ implies that f is a Lattès map. Theorem 1.7 concludes the proof of Theorem 1.1.

While the following is not used in the proof, it follows naturally from the arguments we give; it shows that the tangents arising in the proof can in fact be constructed directly.

Remark 1.9 *A posteriori we note that, in the case of maps, it is possible to construct from a weak tangent at a fixed point the universal orbifold covering map $\pi : \tilde{X} \to X$ associated to the Lattès map together with an expanding map $\psi : \tilde{X} \to \tilde{X}$ whose iterates are uniformly quasisymmetric and such that $\pi \circ \psi = f \circ \pi$.*

1.5 Outline of the paper

The core of the paper deals with the proof when $\mathcal{D}$ is a topologically cxc mapping, since this is new. The case of hyperbolic groups is sketched in the appendix A3. In section 2, we relate Lattès maps to negatively curved homogeneous three-manifolds and prove Theorems 1.7 and 1.8. In section 3, we recall and complete results concerning weak tangents of Ahlfors-regular metrics in the gauge of a topologically cxc map f on S^2. We also prove (Theorem 3.2) that weak tangent spaces of doubling annularly linearly connected surfaces are homeomorphic to the plane. Section 4 is devoted to the notion of moduli of curves and its coarse version of so-called combinatorial moduli. There, we establish the new estimate comparing combinatorial moduli in different dimensions. We conclude this section with the proof of statement (1) in the above dichotomy. Section 5 is devoted to proving that there families of curves of positive modulus, when the sphere is endowed with a metric of minimal dimension. In section 6 we exhibit a foliation by locally thick curves when no thick curve cross: this proves statement (2) in the dichotomy and proves Theorem 1.2 when $\mathcal{D}$ is topologically cxc. In section 7, we summarize results and establish Theorem 1.1. In the appendix, we provide several applications of our estimate on combinatorial moduli of different dimensions and we sketch the proof of Theorem 1.3 by first establishing Theorem 1.2. We also state the results which still hold for general topologically cxc maps.

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1.7 Notation and conventions

Throughout, f denotes an orientation-preserving branched covering map of the two-sphere to itself of degree $d \geq 2$, unless otherwise specified. If X is a metric space, the distance function d on X is understood, and $R > 0$, we denote by RX the metric space $(X, R \cdot d)$. Similarly, if $B = B(x, r)$ is a ball in X and $c > 0$, we set $cB = B(x, cr)$. When convenient we use the notation $|x - y|$ for $d(x, y)$. For two positive functions a, b we write $a \lesssim b$ if $a \leq C \cdot b$; $a \asymp b$ means $a \lesssim b$ and $b \lesssim a$. A sequence whose nth term is a_n is denoted (a_n). The round Euclidean two-sphere of constant curvature $+1$ is denoted $\mathbb{S}^2$.

2 Classification of homogeneous manifolds and of Lattès maps

In this section, we relate Lattès maps to negatively curved homogeneous three-manifolds and give a proof of both Theorems 1.7 and 1.8.

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be an expanding linear map with integer coefficients.

We consider the Lie group G obtained by the extension of the Abelian group $\mathbb{R}^2$ by A, endowed with an invariant Riemannian metric. It follows from Heintze [Hei, Thm. 1] that G is a negatively curved homogeneous manifold.

More precisely, we let B be a matrix such that $\exp B = A$. We consider the three-dimensional Lie algebra $\mathfrak{g}$ such that its derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is Abelian and two-dimensional; we fix a basis (v_1, v_2) of $\mathfrak{g}'$. We also assume that there exists a vector $v_0 \in \mathfrak{g}$ such that, for all $w \in \mathfrak{g}'$, $[v_0, w] = B(w)$, when written in the basis (v_1, v_2). We let $\langle \cdot, \cdot \rangle$ be the scalar product on $\mathfrak{g}$ which makes the basis (v_0, v_1, v_2) orthonormal. This defines a Riemannian metric on G.

Then the pointed boundary $\partial G \setminus \{\infty\}$ can be identified with $\mathfrak{g}' = \mathbb{R}^2$ endowed with a visual metric. The Abelian subgroup $\mathbb{R}^2$ acts freely by isometries in this metric and the vector v_0 via the matrix A induces a dilation $\tilde{f}_A$. The Lebesgue measure of $\mathbb{R}^2$ defines a so-called conformal measure on $\partial G \setminus \{\infty\}$.

Taking a quotient of $\partial G \setminus \{\infty\}$ by the group generated by $\mathbb{Z}^2$ and $-Id$ yields a metric two-sphere. The dilation $\tilde{f}_A$ descends to a topologically cxc map—the associated Lattès map f_A. By Theorem 1.5, the metric belongs to the canonical gauge of f_A, so the Ahlfors-regular conformal dimension of ∂G coincides with the Ahlfors-regular conformal dimension of f_A.

Proof: (Theorems 1.7 and 1.8) When A is semisimple then the Ahlfors-regular conformal dimension of the boundary at infinity of G is attained [Pan]: if there are two real eigenvalues $1 < |\lambda| \le |\mu|$, then $\mathrm{confdim}_{AR}\partial G = 1 + \log|\mu|/\log|\lambda|$. So if the eigenvalues are complex conjugate, then $\mathrm{confdim}_{AR}\partial G = 2$ and is attained. One may find an explicit metric on $\mathbb{R}^2$ of minimal dimension. In the real case, let (v_1, v_2) be a basis by eigenvectors associated to λ and μ respectively. Then set

$$d(x_1 v_1 + x_2 v_2, y_1 v_1 + y_2 v_2) = |y_1 - x_1| + |y_2 - y_1|^\alpha$$

with $\alpha = \log|\lambda|/\log|\mu|$. One may easily check that for this metric, the map $\tilde{f}_A$ acts as a dilation, the corresponding Hausdorff dimension is $1 + 1/\alpha$, and the conformal dimension is minimized. In the complex case, write $A = \lambda \cdot R$ where R is a rotation. This defines canonically a $\mathbb{C}$-linear map, so the Ahlfors-regular conformal dimension is 2.

We may now assume that A is not semisimple. Then A has a double eigenvalue e^λ, $\lambda > 0$. We may then find a basis in $\mathbb{R}^2$ such that

$$B = \begin{pmatrix} \lambda & \delta \\ 0 & \lambda \end{pmatrix}$$

with δ arbitrarily small. Using an explicit computation of the sectional curvatures given in the proof of [Hei, Thm. 1] yields that as $\delta \to 0$ the sectional curvature satisfies $K(x, y) = -\lambda^2 + O(\delta)$ for any orthonormal vectors x, y.

It follows from [Bou] that one may choose the parameter ε of the visual distance to be of order $\lambda + O(\delta)$. Furthermore, the Bishop inequality [GHL, Thm. 3.101] implies that the volume entropy $h(G)$ of G is bounded from above by $2\lambda + O(\delta)$. Finally, [Pan, Lma 5.2] implies that $\mathrm{confdim}_{AR}\partial G \leq 2 + O(\delta)$. Letting δ tend to 0 establishes that $\mathrm{confdim}_{AR}\partial G = 2$.

It remains to prove that this dimension is not attained. Recall that $\tilde{f}_A$ defines a topologically cxc postcritically finite branched covering f_A of the sphere with orbifold $(2, 2, 2, 2)$. The visual distances considered above all belong to the gauge of f_A since $\tilde{f}_A$ acts as a dilation with respect to this metric. If the conformal dimension were attained, then f_A would be conjugate to a rational map by Theorem 1.6. But, according to [DH, Prop. 9.7], the matrix A should then be an integral multiple of the identity. Therefore, the conformal dimension is two but is not attained. This ends the proof of both theorems.

■

3 Weak tangents

We first recall the definitions of the Gromov-Hausdorff topology and of weak tangent spaces. We then prove that weak tangents of ALC surfaces are homeomorphic to $\mathbb{R}^2$, and we establish properties which will be used in the sequel.

3.1 Gromov-Hausdorff convergence

Let Z be a proper metric space. Given two subsets $X, Y \subset Z$, define their Hausdorff distance as

$$d_H(X, Y) = \max\left\{\sup_{x\in X} d(x, Y), \sup_{y\in Y} d(y, X)\right\}.$$

Note that $d_H(X, Y) = d_H(\overline{X}, \overline{Y})$. The set of nonempty compact subsets of Z endowed with d_H is complete and is sequentially compact if Z is compact.

Given two compact metric spaces X and Y,we define their *Gromov-Hausdorff distance* by

$$d_{HG}(X, Y) = \inf_Z\{d_H(X, Y),\ X, Y \subset Z\}$$

where the infimum is taken over all compact metric spaces Z which contains isometric copies of both X and Y. This defines a distance function on the set of isometry classes of compact metric spaces.

In this topology, a sequence (X_n) of compact metric spaces is sequentially relatively compact if and only if the sequence is *uniformly compact* i.e., for any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that each set X_n can be covered by $N(\varepsilon)$ balls of radius ε.

For a uniformly compact family of metric spaces $(X_\alpha)_{\alpha\in A}$, there exists a compact metric space Z and isometric embeddings $X_\alpha \hookrightarrow Z$. It follows that a sequence (X_n) of compact metric spaces converges to X with respect to the Gromov-Hausdorff distance if and only if there exists a fixed compact metric space Z and isometric embeddings $j_n : X_n \hookrightarrow Z$ and $j : X \hookrightarrow Z$ such that $j_n(X_n) \to j(X)$ in the Hausdorff topology on compact subsets of Z. Henceforth, we will regard the X_n and the limit X simply as subsets of Z. The completeness of the Gromov-Hausdorff metric implies the limit is in fact independent of Z.

We now extend the notion of convergence to noncompact pointed spaces. Say a sequence of pointed proper metric spaces (X_n, x_n) *converges in the Gromov-Hausdorff topology* to a pointed metric space (X, x) if, for all radii $R > 0$, the sequence of compact pointed closed balls $(\overline{B_{X_n}(x_n, R)}, x_n)$ in X_n converges with respect to the Gromov-Hausdorff distance to a compact set B_R in X which contains the open ball $(B_X(x, R), x)$.

Finally, we extend the notion of convergence to subsets of pointed noncompact spaces. Suppose a sequence of pointed proper metric spaces (X_n, x_n) converges. By definition, this implies that for all $R > 0$, the sequence of compact closed balls $\overline{B_{X_n}(x_n, R)}$ converges. This sequence must be uniformly compact, so there is a fixed compact metric space Z_R into which each closed ball $\overline{B_{X_n}(x_n, r)}$ embeds. When $R_1 < R_2$ we may assume $Z_{R_1} \hookrightarrow Z_{R_2}$ isometrically and so we obtain an inductive system of pointed proper metric spaces $(Z_R)_{R>0}$ containing embedded isometric copies of $(\overline{B(x_n, R)})_n$.

Definition 3.1 *Suppose (X_n, x_n) is a convergent sequence of pointed proper metric spaces, and $Y_n \subset X_n$ are subsets. The sequence Y_n is said to* converge to Y *if*

1. *there exists $R_0 > 0$ such that $Y_n \cap B_{X_n}(x_n, R_0) \neq \emptyset$ for all n sufficiently large, and*
2. *for all $R \geq R_0$, the compact sets $\overline{Y_n} \cap \overline{B_{X_n}(x_n, R)}$ converge in the Hausdorff topology on compact subsets of Z_R.*

Convergence of maps is defined in the obvious way: suppose $(X_n, x_n) \to (X, x)$, $(Y_n, y_n) \to (Y, y)$, $f_n : (X_n, x_n) \to (Y_n, y_n)$, and $f : (X, x) \to (Y, y)$. We say that $f_n \to f$ *in the Gromov-Hausdorff topology* if, after identifying spaces with their images under the above embeddings, we have $f_n(x_n) \to f(x)$ for every sequence (x_n) with $x_n \in X_n$ for each $n \in \mathbb{N}$, such that $x_n \to x$.

3.2 Weak tangents

The notion of weak tangent formalizes the processes of zooming in near a point and passing to a limit. This requires some tameness on the metric space. A proper metric space X with distance function d is *N-doubling*, or *doubling* if N bears no particular interest, if any ball of finite radius can be covered by N balls of half its radius. In this case the family $\{(X, x, Rd)\}_{x\in X, R>0}$ is relatively compact in the Gromov-Hausdorff topology. A limit point of a sequence $(X, x_n, R_n d)$ of pointed rescaled spaces with $R_n \to \infty$ and (x_n) lying in a compact subset of X is called a *weak tangent* of X. We speak of *weak tangents at* $x_0 \in X$ if it is a limit of the sequence $(X, x_0, R_n d)$, i.e. $x_n = x_0$ is a constant sequence.

Note that any limit will be at most $2N$-doubling.

One may also consider limits of metric measure spaces (X_n, x_n, μ_n) where the μ_n's are Radon measures. This means that (X_n, x_n) converges as a sequence of metric spaces, and in addition there exists constants c_n such that $c_n \cdot \mu_n$ weakly converges. In particular, if (μ_n) are all Ahlfors-regular measures of the same dimension and with uniform constants, then, rescaling the measures if needed, one may extract a convergent subsequence to an Ahlfors regular measure μ in the limit.

3.3 Limits of surfaces

The main result of this subsection is the following:

Theorem 3.2 *Let X be a doubling proper metric surface which is ALC. Then any weak tangent space T is ALC and is homeomorphic to the plane.*

The doubling property is used to ensure the existence of a tangent space, even if this is not a necessary condition.

The proof has several steps.

1. Since the ALC condition is scale-independent, a straightforward argument shows T is ALC.

2. The definition of ALC implies immediately that the one-point, or Alexandroff, compactification $\hat{T}$ of T is a locally connected continuum without local cut points.

3. Next, we prove that $\hat{T}$ is embeddable in the sphere. To do this, we use the planarity of X and a stability result regarding limits of graphs (Proposition 3.6) to show that $\hat{T}$ cannot contain certain non-planar graphs. We then appeal to a classical characterization theorem of Claytor [Cla].

4. Therefore, the complement of $\hat{T}$ in the sphere is either empty, or consists of Jordan domains. The remainder of the proof consists of ruling out this latter possibility. We do this by analyzing the complement of simple closed curves in T.

Our graph stability result depends on three technical lemmas.

The first one says that tripods can be unzipped to a pair of arcs meeting at a point in such a way that the modification takes place only near two of the arms. Think of U as a small neighborhood of γ'. The arc α might enter and exit U many times.

Lemma 3.3 *Let $\gamma' : [0,1] \to X$ be an arc in an L-ALC metric space X and let U be an arcwise connected open set which contains $\gamma'([0,1))$. Let $\alpha' : [0,1] \to X$ be an arc with the following properties: $\alpha'(0) \notin U$, $\alpha'([0,1)) \cap \gamma' = \emptyset$ and $\alpha'(1) \in (\gamma' \cap U)$. Then there are arcs $\alpha : [0,1] \to X$ and $\gamma : [0,1] \to X$ such that*

1. *(a)* $\gamma([0,1)) \subset U$,

   (b) $\gamma(1) = \gamma'(1)$,

   (c) there exists $0 < s_0 \leq 1$ *such that* $\gamma|_{[0,s_0]} = \gamma'|_{[0,s_0]}$;

2. *there exists* $0 < t_0 < 1$ *such that*

   (a) $\alpha|_{[0,t_0]} = \alpha'|_{[0,t_0]}$,

   (b) $\alpha([t_0,1)) \subset U$,

   (c) $\alpha([t_0,1)) \cap \gamma([0,1)) = \emptyset$,

   (d) $\alpha(1) = \alpha'(1) = \gamma'(1) = \gamma(1)$.

Proof: We will follow the same strategy as [Mac2, Lma 3.1]. We will "unzip" the tripod defined by γ' and α' to have two curves which only meet at $x_0 = \gamma'(1)$.

Let x be the intersection point between γ' and α'. Let $r_x = \min\{|x - x_0|, d(x, X \setminus U)\}/L$. Pick a point $y \in \gamma' \cap (B(x, r_x) \setminus B(x, r_x/2))$ beyond x, and a point $y' \in \alpha' \cap (B(x, r_x) \setminus B(x, r_x/2))$ before x. By (ALC2), there is an arc A in U which joins y' to y in $B(x, Lr_x) \setminus B(x, r_x/(2L))$. If A meets $\gamma' \cup \alpha'$ only at its extremities y and y', use A to replace α' beyond y'. Otherwise, let z be the first point in γ' encountered beyond x by A, and z' the first point before z in A which meets $\gamma' \cup \alpha'$: if z' is on α', replace the piece of α' between z' and x by A, which now ends at z; if z' is on γ', replace the part of α' between z' and z by A, and add to α' the piece of γ' between x and z.

This implies that we have now two curves γ and α which meet at a definitely closer point from x_0 than before, and they both continue with γ' up to x_0; moreover, they are contained in U. We continue inductively. At each stage, modifications only alter the positions in U. Since r_x is locally bounded from below for $x \neq x_0$, both curves meet in the limit exactly at x_0.

■

The next lemma says that a collection of arcs whose endpoints are close can be extended to a multi-armed tripod.

Lemma 3.4 *Let U be an arcwise connected open set of an annularly linearly connected metric space. Let n be a positive integer, and let us consider n pairwise disjoint arcs γ'_j, $j = 1, \ldots, n$, with $\gamma'_j(0) \notin U$, and $\gamma'_j(1) = x_j \in U$. Let $x_0 \in U$ be disjoint from these curves. There are n arcs $\gamma_1, \ldots, \gamma_n$ such that $\gamma_j = \gamma'_j$ off of U and which only meet at their other extremities at x_0.*

Proof: We will procede by induction. Let us assume that we have already constructed $\gamma_1, \ldots, \gamma_{k-1}$ such that the conclusion of the lemma holds.

If γ'_k intersects $\cup\gamma_j$, we let γ be the one which is first met. We may then apply Lemma 3.3 to γ and γ'_k with the connected component of $\{z \in U,\ d(z,\gamma) < d(z,(\cup\gamma_j)\setminus\gamma)\}$ which contains $\gamma\cap U$. This leads to k arcs which satisfy the conclusion of the lemma.

Otherwise, we consider an arc which joins $\gamma'_k(1)$ to x_0 in U. If it does not meet the requirements, we may apply the argument above replacing γ'_k with its extension to the first point of intersection with the other curves.

■

Hausdorff convergence is quite weak. If $X_n \to X$, arcs in X need not be limits of arcs in X_n, even if the X_n are required to be arcwise connected. The lemma below says that if the geometry of the X_n is controlled, then this is possible.

Lemma 3.5 *Suppose X is a compact subset of a proper metric space Z, $\gamma : [0,1] \to X$ is an arc, $\eta > 0$, and $L \geq 1$. Then there exists $\varepsilon > 0$ such that if $d_H(X,Y) \leq \varepsilon$ and Y has L-bounded turning (BT), then, for all $y \in B(\gamma(0),\varepsilon)\cap Y$ and $y' \in B(\gamma(1),\varepsilon)\cap Y$ there exists an arc $c : [0,1] \to Y$ such that $c(0) = y$, $c(1) = y'$ and*

$$\sup_{t\in[0,1]} |c(t)-\gamma(t)| \leq \eta\,.$$

Proof: Set $\varepsilon = d_H(X,Y)$. We will construct an arc $c = c_\varepsilon$ in Y such that

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,1]} |c_\varepsilon(t)-\gamma(t)| = 0\,.$$

We first record some facts. Since γ is an arc, there are increasing homeomorphisms $\omega_\pm : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $s,t \in [0,1]$,

$$\omega_-(|s-t|) \leq |\gamma(s)-\gamma(t)| \leq \omega_+(|s-t|)\,.$$

We note that, for any $0 \leq s < t \leq 1$, one can find an arc $c : [s,t] \to Y$ such that

$$\sup_{u\in[s,t]} |c(u)-\gamma(u)| \leq \varepsilon + L(\omega_+(|s-t|)+2\varepsilon) + \omega_+(|s-t|)\,. \tag{1}$$

To prove this, pick $c(s), c(t)$ in Y at distance at most ε from $\gamma(s), \gamma(t)$ respectively. By the (BT) property, there exists an arc $c : [s,t] \to Y$ such that

$$\operatorname{diam} c([s,t]) \leq L|c(s)-c(t)| \leq L(\omega_+(|s-t|)+2\varepsilon)\,. \tag{2}$$

Therefore, for any $u \in [s,t]$, the triangle inequality shows that

$$\begin{aligned} |c(u)-\gamma(u)| &\leq |c(u)-c(s)| + |c(s)-\gamma(s)| + |\gamma(s)-\gamma(u)| \\ &\leq L(\omega_+(|s-t|)+2\varepsilon) + \varepsilon + \omega_+(|s-t|)\,. \end{aligned}$$

Fix an integer $n \geq 1$ so that $\omega_+(1/n) \leq \varepsilon$. For $j = 0, \ldots n$, let $t_j = j/n$, $x_j = \gamma(t_j)$, and consider $y_j \in Y$ such that $|x_j - y_j| \leq \varepsilon$ with $y_0 = y$ and $y_n = y'$. By the procedure above, build for $j = 0, \ldots, (n-1)$, an arc $c_j : [t_j, t_{j+1}] \to Y$ joining y_j to y_{j+1} such that, according to (1), for any $t \in [t_j, t_{j+1}]$,

$$|c_j(t) - \gamma(t)| \leq (3L + 2)\varepsilon . \tag{3}$$

Note that if $c_j \cap c_k \neq \emptyset$ for $j < k$, then, by (2),

$$|y_j - y_{k+1}| \leq \operatorname{diam} c_j + \operatorname{diam} c_k \leq 6L\varepsilon ,$$

so that $|x_j - x_{k+1}| \leq (6L+2)\varepsilon$. But since $|x_j - x_{k+1}| \geq \omega_-(t_{k+1} - t_j)$, it follows that

$$|t_{k+1} - t_j| \leq \omega_-^{-1}((6L+2)\varepsilon) .$$

Therefore, for any $t \in [t_j, t_{k+1}]$,

$$|\gamma(t) - x_j| \leq (\omega_+ \circ \omega_-^{-1})((6L+2)\varepsilon) . \tag{4}$$

We extract an arc $c : [0,1] \to Y$ from the c_j's by induction as follows. Let us first define $\kappa : [0,1] \to Y$ as the concatenation of the arcs c_j's. Set $s_0 = 0$ and $c(s_0) = \kappa(0)$. If s_{j-1} and $c|_{[0,s_{j-1}]}$ are constructed, let

$$u_j = \min\{t \in [s_{j-1}, 1],\ \exists\, s > t,\ \kappa(t) = \kappa(s)\}$$

if it exists or $u_j = 1$ otherwise, and

$$s_j = \max\{t \in [0,1],\ \kappa(t) = \kappa(u_j)\} .$$

If $u_j = 1$, then we let $c|_{[s_{j-1},1]} = \kappa|_{[s_{j-1},1]}$. Otherwise, set $c|_{[s_{j-1},u_j]} = \kappa|_{[s_{j-1},u_j]}$ and $c|_{[u_j,s_j]} = \kappa(u_j)$. We may then continue. Since each c_j is an arc, this procedure stops after at most n steps. We obtain a parametrized arc c with connected fibers.

We now estimate the sup-norm. If $t \in [s_j, u_{j+1}]$ for some j, then $c(t)$ coincides with $\kappa(t)$ so (3) implies

$$|c(t) - \gamma(t)| \leq (3L+2)\varepsilon .$$

If $t \in [u_j, s_j]$, then $c(t) = \kappa(u_j)$ and there is some index $0 \leq k < n$ such that $t_k \leq u_j < t_{k+1}$. Applying (2) and (4), we obtain

$$\begin{aligned} |c(t) - \gamma(t)| &\leq |c_k(u_j) - y_k| + |y_k - x_k| + |x_k - \gamma(t)| \\ &\leq 3L\varepsilon + \varepsilon + (\omega_+ \circ \omega_-^{-1})((3L+2)\varepsilon) . \end{aligned}$$

■

Proposition 3.6 *Let X be an L-annularly linearly connected doubling proper metric space. Let T be a limit of $X_n = (X, x_n, R_n d)$ for $x_n \in X$ and $R_n \to \infty$. Suppose $f : \Gamma \to \hat{T}$ is an embedding of Γ. Then for all n sufficiently large, there exists an embedding $f_n : \Gamma \to X_n$. Moreover, if $f(\Gamma) \subset T$, then the f_n may be chosen so that $\sup_{t\in\Gamma} |f_n(t) - f(t)| \to 0$ as $n \to \infty$.*

Proof: If B is a ball in an ALC metric space, we let B^0 denote its connected component which contains the center of the ball, so that $(1/L)B \subset B^0 \subset B$.

Assume first that $f(\Gamma) \subset T$. Then $f(\Gamma)$ is compact so we may replace the spaces X_n, T with closed balls, and we may assume the convergence of spaces is Hausdorff convergence in a compact metric space Z. It is convenient to supply Γ with a length distance which makes every edge isometric to $[0, 1]$. We may think of the restriction of f to each edge e as given by a map $f_e : [0, 1] \to \Gamma$. Let V be the vertex set of Γ. Choose $r > 0$ so that $|f(v) - f(w)| \geq 6Lr$ for each pair of distinct vertices v, w.

The continuity of f implies the existence of $\delta > 0$ such that if $x \in \Gamma$ and $d(x, V) \leq \delta$ then $d(f(x), f(V)) \leq r/2$. Furthermore, since f is a homeomorphism onto its image, there is some $\eta \in (0, r/2)$ such that, for any $(x, x') \in e \times e'$ with $e \neq e'$, if $|f(x) - f(x')| \leq \eta$ then $d(x, V) \leq \delta$. Since X is ALC, the spaces X_n have uniformly bounded turning. For each n and each vertex $v \in V$, we pick points $x_v^n \in X_n$ such that $|f(v) - x_v^n| \leq d_H(T, X_n)$. According to Lemma 3.5, there is some n_0 such that, for each $n \geq n_0$, for each edge e, there is an arc $c_e^n : [0, 1] \to X_n$ approximating the edge f_e with $c_e^n(v) = x_v^n$ and $\sup |c_e^n(t) - f_e(t)| \leq \eta/2$. We let $f_n : \Gamma \to X_n$ be the map obtained by setting $f_n|_e = c_e^n$, i.e. by concatenating each of the edge approximations.

If f_n is injective, then we are done. Otherwise, if $f_n(x) = f_n(x')$ for some $x \neq x'$, then x and x' belong to different edges and $|f(x) - f(x')| \leq |f(x) - f_n(x)| + |f_n(x') - f(x')| \leq \eta$; therefore, $d(x, V) \leq \delta$, $d(f(x), f(V)) \leq r/2$ and $d(f_n(x), x_v^n) \leq \eta + r/2 \leq r$. Hence, f_n is injective over $X_n \setminus (\cup_v B^0(x_v^n, Lr))$. For each $v \in V$, let $e_1, \ldots, e_m$ be the edges incident to v, and assume that $f_{e_j}(0) = v$ for all j. Apply Lemma 3.4 with $U = B^0(x_v^n, 2Lr)$ to truncated edges (so that they are disjoint) to obtain an extension of the collection of arcs $\{c_{e_j}^n\}_{j=1}^m$ to arcs $[0, 1] \to X_n$ meeting only at x_v^n. We have therefore produced an embedding $f_n : \Gamma \to X_n$. Note that if $f_n(x) \in B^0(x_v^n, 2Lr)$, then $|f_n(x) - f(x)| \leq 4Lr + \eta/2 \leq 5Lr$. Hence f_n and f are $5Lr$-close.

Now suppose $f(\Gamma)$ meets the point at infinity. We may assume the point at infinity is the image of a vertex v_∞. By bisecting the edges meeting the point at infinity we may assume there is a distinguished collection of valence two vertices $v_1, \ldots, v_m$ comprising the ends of a star-shaped graph with center at v_∞. Suppose the edge e_j joins the vertex v_j to v_∞, and the edge g_j is the other edge incident to v_j, $j = 1, \ldots, m$. Let Γ' be the subgraph obtained by deleting the edges $e_1, \ldots, e_m$, and let Γ'' be the subgraph obtained by deleting the edges $e_1, \ldots, e_m$ and the edges $g_1, \ldots, g_m$.

By choosing $v_1, \dots, v_m$ close enough to v_∞, we may assume that $f(\Gamma'')$ is contained in a ball B and that the images $f(v_1), \dots, f(v_m)$ are contained in the ball $100LB$. By the first case, the restriction $f : \Gamma' \to T$ is approximated by $f_n : \Gamma' \to X_n$. Let B_n be a ball in X_n which is close to B; let U_n be the unbounded connected component of $X_n \setminus \overline{B_n}$. By the choice of B, the set U_n is arcwise connected and contains the points $f_n(v_1), \dots, f_n(v_n)$, which are endpoints of arcs g_j^n approximating the arcs g_j. Choose arbitrarily a point $x_n^\infty \in U_n$ which does not meet the image of any arc g_j^n. By Lemma 3.4, there is an extension of the arcs g_j^n to a collection of arcs which meet only at x_n^∞. We have produced an embedding of Γ into X_n and the proof is complete.

■

Remark 3.7 *The proof shows that if Z_n are compact and uniformly ALC and if $Z_n \to Z$ in the Gromov-Hausdorff topology, then any finite graph embedding $f : \Gamma \to Z$ is a limit of embeddings $f_n : \Gamma \to Z_n$.*

We now prove the theorem. We first record some facts and notation. Let X be an unbounded ALC proper metric space. Given any compact set K and a point $x \in K$, there is a single connected component of $X \setminus K$ which contains points from $X \setminus B(x, 2L\mathrm{diam}K)$. Therefore the *filling-in* of K consisting of K with the other components of $X \setminus K$ is a compact subset of $B(x, 2L\mathrm{diam}K)$.

Proof: (Thm 3.2). Let (R_n) tend to $+\infty$ and (T, t, d_T) be a limit of $X_n = (X, x, R_n d)$. We may assume that the spaces are homeomorphic to planes for the statement is local. Since the Aleksandroff compactification $\hat{T}$ of T is locally connected with no local cut points, Claytor's imbedding theorem implies that it is embeddable in the sphere if $\hat{T}$ has no embbeded complete graphs on five vertices, nor a complete bipartite graph on three vertices [Cla]. This is the content of Proposition 3.6: if there were such a graph, then X_n would also have a copy in a Jordan neighborhood of x, which is impossible.

This implies that we may think of $\hat{T}$ as a locally connected continuum of the sphere with no local cut points. If this is the sphere then the proof is complete. Otherwise, each connected component of the complement is a Jordan domain, see [Why, Thm. VI.4.4].

Claim.— *For any simply closed curve γ in T, there are two continua of $\hat{T}$, Ω and D, with $\Omega \cap T$ unbounded in T and D bounded, such that $\Omega \cap D = \gamma$, $\Omega \cup D = \hat{T}$ and, for any $z \in \gamma$ and any $r > 0$, both intersections $(\Omega \setminus \gamma) \cap B(z, r)$ and $(D \setminus \gamma) \cap B(z, r)$ are nonempty.*

The claim implies that $S^2 \setminus \hat{T}$ has no bounded components. Since the point at infinity is not a local cut point of $\hat{T}$, there can be at most one unbounded component, so that $\hat{T}$ is a closed disk. Applying the Claim to a bounded Jordan curve γ in T which follows a piece of its boundary, the claim also leads us to a contradiction. So, up to the claim, the theorem is proven.

Proof of the claim. — Let γ be a simple closed curve in T. Choosing arbitrarily a pair of vertices, we may regard it as a graph. Proposition 3.6 implies that γ is a limit of parametrized Jordan curves γ_n in X_n where the parametrizations converge uniformly. Each γ_n for n large enough separates X_n into two components D_n and Ω_n, the latter being unbounded. We may assume that $\overline{D}_n \to D$ and $\overline{\Omega}_n \to \Omega$ where D, Ω are closed connected subsets of T. Adding the point at infinity to Ω provides us with a covering of $\hat{T}$ by two continua.

In this paragraph, we prove that $\gamma = \Omega \cap D$. Suppose first that $z \in \Omega \cap D$. Then by definition, there exist $w_n \in D_n$, $w_n' \in \Omega_n$ with $w_n \to z$ and $w_n' \to z$ as $n \to \infty$. The (BT) property implies that there is a curve c_n joining w_n to w_n' such that $\operatorname{diam} c_n \le L \cdot d_n(w_n, w_n')$. The Jordan curve theorem implies that c_n intersects γ_n in a point z_n. As n tends to infinity, $z_n \to z$, so we obtain that $z \in \gamma$. Conversely, suppose now $z \in \gamma$. Then $z = \lim z_n$ with $z_n \in \gamma_n$. Applying the Jordan curve theorem again gives $w_n \in D_n$ and $w_n' \in \Omega_n$ arbitrarily close to z_n; we may therefore assume $w_n \to z$ and $w_n' \to z$ and so $z \in \Omega \cap D$.

For the second part of the claim, we argue by contradiction. Suppose one of the intersections in the statement is empty; we will only treat the case of D. So, assume that there is some $z \in \gamma$ and $r > 0$ such that $D \cap B(z, r) \subset \gamma$. Fix $0 < \varepsilon \ll r$; we will prove that this implies the existence of points $z_n, z_n' \in \gamma_n$ such that the connected components of $\gamma_n \setminus \{z_n, z_n'\}$ have diameter comparable to that of γ_n but that $d_n(z_n, z_n') \le 2\varepsilon$. Letting ε go to 0 and going to the limit, this will contradict that γ is a simple closed curve.

Fix a point y on γ at distance $\frac{1}{2}\operatorname{diam}\gamma$ from z. Suppose $y_n, z_n \in \gamma_n$ and $y_n \to y$ and $z_n \to z$. By assumption, for n large enough, $D_n \cap B(z_n, r/2)$ is contained in the ε-neighborhood of γ_n.

Let E_n be the closure of the filling-in of the component of $B(z_n, r/6L^2)$ containing z_n and F_n be the closure of the unbounded component of $X_n \setminus B(z_n, r/2L)$ so that $d(E_n, F_n) \ge (r/3L)$ and $A_n \subset B(z_n, r/2)$ hold. We may assume that r is small enough so that F_n contains y_n. The set $A_n = X_n \setminus (E_n \cup F_n)$ is an annulus, and the curves γ_n must connect the ends (and may also connect an end with itself); see Figure 1. Denote by c_1^n and c_2^n the closures of the two components of $\gamma_n \setminus \{z_n, y_n\}$; they are given by parametrizations (which we denote also by c_j^n) defined on compact intervals. By the uniform continuity of the c_j^n, there is some $\delta > 0$ such that if $|s - t| \le \delta$ then $|c_j^n(s) - c_j^n(t)| \le d(E_n, F_n)/2$. Since the c_j^n are defined on compact intervals, it follows that there are only finitely many components of the intersection $c_j^n \cap A_n$ joining the end E_n to the end F_n. Hence there is a component U_n of $A_n \cap D_n$ which has in its boundary pieces of both c_1^n and of c_2^n. By assumption, $D_n \cap B(z_n, r/2)$ is contained in the ε-neighborhood of γ_n, and $A_n \subset B(z_n, r/2)$, so the component U_n of the intersection is contained in the ε-neighborhood of γ_n.

It follows that for all $u \in U_n$, we have that $d(u, \partial U_n) \le d(u, \gamma_n \cap \partial U_n) \le \varepsilon$ where $d(u, \partial U_n)$ is the minimum distance from u to a point in ∂U. For each $j = 1, 2$, the set U_n^j of points in U_n which are at distance strictly less than ε from c_j^n is non-empty and open, hence the connectivity of U_n implies that at least one point in U_n is at distance at most ε from both curves c_1^n and c_2^n: we may find $w_j^n \in c_j^n$ such that $d_n(w_1^n, w_2^n) < 2\varepsilon$. Furthermore, the diameters of the components of $\gamma_n \setminus \{w_1, w_2\}$ are bounded from below by a constant which depends only on r and $\operatorname{diam}\gamma_n$. This ends the proofs of the claim and of

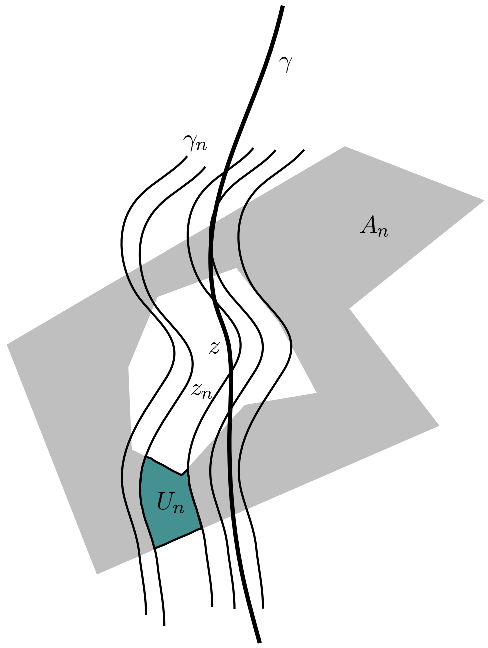


Figure 1

the theorem.

■

3.4 Topological mixing properties

We will assume throughout this subsection that f is a topologically cxc map from the sphere to itself, endowed with a metric d_v as in Theorem 1.5. The main result, Proposition 3.9, articulates in quantitative form the principle of the Conformal Elevator: arbitrarily small balls can be blown up via the dynamics with controlled distortion. The proof will use the fact ([HP1, Cor. 3.5.4]) that the map f is a so-called metric cxc map with respect to (S^2, d_v). Before recalling the properties needed, we define the notion of roundness.

Roundness. Let (Z, d) be a metric space and let A be a bounded, proper subset of Z with nonempty interior. Given $a \in \text{int}(A)$, let

$$L(A, a) = \sup\{d(a, b) : b \in A\}$$

and

$$l(A, a) = \sup\{r : r \leq L(A, a) \text{ and } B(a, r) \subset A\}$$

denote, respectively, the *outradius* and *inradius* of A about a. While the outradius is intrinsic, the inradius depends on how A sits in Z. The condition $r \leq L(A, a)$ is necessary to guarantee that the outradius is at least the inradius. The *roundness of A about a* is defined as

$$\mathrm{Round}(A,a) = L(A,a)/l(A,a) \in [1,\infty).$$

One says A is *K-almost-round* if $\mathrm{Round}(A,a) \leq K$ for some $a \in A$, and this implies that for some $s > 0$,

$$B(a,s) \subset A \subset B(a,Ks).$$

Being metric cxc implies the existence of

- continuous, increasing embeddings $\rho_\pm : [1,\infty) \to [1,\infty)$ such that, for all $n,k \in \mathbb{N}$ and for all $\widetilde{U} \in \mathcal{U}_{n+k}$, $\tilde{y} \in \widetilde{U}$, if $U = f^k(\widetilde{U}) \in \mathcal{U}_n$ and $y = f^k(\tilde{y})$ then

$$\mathrm{Round}(\widetilde{U},\tilde{y}) < \rho_-(\mathrm{Round}(U,y)) \quad \text{and} \quad \mathrm{Round}(U,y) < \rho_+(\mathrm{Round}(\widetilde{U},\tilde{y}))\,;$$

- increasing homeomorphisms $\delta_\pm : [0,1] \to [0,1]$ such that, for all $n_0, n_1, k \in \mathbb{N}$ and for all $\widetilde{U} \in \mathcal{U}_{n_0+k}$, $\widetilde{U}' \in \mathcal{U}_{n_1+k}$ with $\widetilde{U}' \subset \widetilde{U}$, if $U = f^k(\widetilde{U}) \in \mathcal{U}_{n_0}$ and $U' =, f^k(\widetilde{U}') \in \mathcal{U}_{n_1}$, then

$$\delta_+^{-1}\left(\frac{\mathrm{diam}\, U'}{\mathrm{diam}\, U}\right) \leq \frac{\mathrm{diam}\widetilde{U}'}{\mathrm{diam}\widetilde{U}} \leq \delta_-\left(\frac{\mathrm{diam}U'}{\mathrm{diam}U}\right).$$

To control distortion, some "Koebe space" is needed. To this end, let $\mathcal{P}_{k,K}$ denote the set of *preferred pairs* (W',W) of elements of $\mathbf{U}$ such that $|W'| = |W| + k$, $W' \subset W$ and $\mathrm{Round}(W,x) \leq K$ for any $x \in W'$.

Proposition 3.8 *1. For any $K \geq 1$, there exists K' such that, whenever $(U',U) \in \mathcal{P}_{k,K}$, $W', W \in \mathbf{U}$ are such that $f^p(U') = f^q(W')$ and $f^p(U) = f^q(W)$ belong to $\mathbf{U}$ for some iterates p and q, and $W' \subset W$, then $(W',W) \in \mathcal{P}_{k,K'}$.*

2. There exist n_1 and K such that, for any $k_j \in \mathbb{N}$, any $U_j \in \mathcal{U}_{k_j}$, $j = 1,2$, there exists $\widetilde{U}_1 \in f^{-(k_2+n_1)}(\{U_1\})$ such that $(\widetilde{U}_1, U_2) \in \mathcal{P}_{k_1+n_1,K}$.

3. There exists k_0 and K_0 such that, for any $n \geq 0$ and any $k \geq k_0$,

(a) for any $W' \in \mathcal{U}_{n+k}$, there exists $W \in \mathcal{U}_n$ such that $(W',W) \in \mathcal{P}_{k,K_0}$;

(b) for any $W \in \mathcal{U}_n$, there exists $W' \in \mathcal{U}_{n+k}$ such that $(W',W) \in \mathcal{P}_{k,K_0}$.

Proof: The first statement follows from the roundness distortion bounds enjoyed by metric cxc maps, by choosing $K' = (\rho_- \circ \rho_+)(K)$. The conclusion in case (a) of the last statement follows directly from the diameter and roundness bounds using the fact that $\mathcal{U}$ is finite; case (b) follows from the second point.

We now prove the second point, and start by fixing some notation. There exists $r_i > 0$ such that, for any $U \in \mathcal{U}$, there is some $x_U \in U$ with $B(x_U, r_i) \subset U$. Let $n_0 \geq 1$ be such that $f^{n_0}(U \cap X) = X$ for any $U \in \mathcal{U}$. Choose $m \geq n_0$ as small as possible such that $2d_m \leq r_i/2$.

We first assume that $U_2 \in \mathcal{U}$. Let $W \in \mathcal{U}_m$ contain x_{U_2}. Since $f^{2m}(W) = X$, there is a component $\widetilde{U}_1$ of $f^{-2m}(U_1)$ which intersects W. It follows that $B(y, r_i - 2d_m) \subset U_2$ for all $y \in \widetilde{U}_1$ since, for all $z \in B(y, r_i - 2d_m)$,

$$\begin{aligned} |x_{U_2} - z| &\leq |x_{U_2} - y| + |y - z| \\ &< \operatorname{diam} W + \operatorname{diam} \widetilde{U}_1 + (r_i - 2d_m) \\ &\leq r_i. \end{aligned}$$

Therefore, $\mathrm{Round}(U_2, y) \leq 2d_0/r_i$. Thus, $(\widetilde{U}_1, U_2) \in \mathcal{P}_{2m+k_1, 2d_0/r_i}$.

Pick now U_2 randomly. It follows from above that there exists $\widetilde{U}_1' \in \mathcal{U}_{2m+k_1}$ such that $(\widetilde{U}_1', f^{k_2}(U_2)) \in \mathcal{P}_{2m+k_1, 2d_0/r_i}$. By choosing a connected component $\widetilde{U}_1$ of $f^{-k_2}(\widetilde{U}_1)$ in U_2, we obtain $(\widetilde{U}_1, U_2) \in \mathcal{P}_{n_1+k_1, K}$ with $n_1 = 2m$ and $K = \rho_-(2d_0/r_i)$.

■

We derive the following property:

Corollary 3.9 (Injective conformal elevator) *For any $X \in \mathcal{G}(f)$, there exist a distortion function η_{ice}, constants $c > 0$ and $r_0 > 0$ such that, for any $x \in X$ and $r > 0$, there are an iterate $n \geq 0$ and a ball $B \subset B(x, r)$ of radius at least $c \cdot r$ such that $f^n|_B$ is η_{ice}-quasisymmetric and $f^n(B)$ contains a ball of radius at least r_0.*

Proof: It is enough to prove the statement for the metric d_v. Recall that $\mathbf{U}$ denotes the countable set of components of preimages of elements of $\mathcal{U}_0$ under iterates of f. The degree hypothesis in the definition of topologically cxc implies that we can find some $W \in \mathcal{U}_k$ such that the degree of $f^k : W \to f^k(W)$ is maximal, so that any further preimages $\widetilde{W}$ of W map onto W by degree one i.e., are homeomorphisms.

It follows from [HP1, Prop. 3.3.2] that W contains some ball $B(\xi, 4R)$ such that for any iterate n, any $\tilde{\xi} \in f^{-n}(\xi)$, the restriction $f^n : B(\tilde{\xi}, 4R\theta^n) \to B(\xi, 4R)$ is a homeomorphism. [HP1, Prop. 3.2.3] shows that $f^n : B(\tilde{\xi}, R\theta^n) \to B(\xi, R)$ is a similarity (in particular, it is quasisymmetric) and that the diameter of the unique preimage $\widetilde{W}$ of W containing $B(\tilde{\xi}, R\theta^n)$ is comparable to $R\theta^n$.

Now suppose $B(x, r)$ is an arbitrary ball. By [HP1, Prop. 2.6.6], there exists $U \in \mathbf{U}$ with $U \subset B(x, r)$ and $\operatorname{diam} U \gtrsim r$. But Proposition 3.8 implies the existence of an iterated preimage $\widetilde{W}$ of W with $\widetilde{W} \subset U$ and $||\widetilde{W}| - |U|| = O(1)$. So U contains $\widetilde{W}$, which is not too small; the previous paragraph implies $\widetilde{W}$ contains the desired ball B.

■

3.5 Density points

Let (Z, d, μ) be a Q-regular metric space. We recall that μ extends to an outer measure (which we also denote by μ) on the power set of Z so that if A is any subset of Z, then there exists a Borel set A^* containing A such that $\mu(A) = \mu(A^*)$.

A point $a \in A$ is an *m-density point* if

$$\lim_{r\to 0} \frac{\mu(A \cap B(a,r))}{\mu(B(a,r))} = 1\,.$$

A point $a \in A$ is a *t-density point* if, for all $\varepsilon > 0$,

$$\lim_{r\to 0} \frac{1}{r} \sup_{z\in B(a,r)} d(z, A \cap B(a,(1+\varepsilon)r)) = 0\,.$$

Informally: the set A becomes hairier and hairier upon zooming in at a t-density point.

Lemma 3.10 *Suppose Z is doubling and $A \subset Z$. Then each m-density point of A is also a t-density point.*

Proof: If not, there are an m-density point a of A, positive constants ε and $c > 0$, sequences of radii (r_n) tending to 0 and of points $z_n \in B(a, r_n)$ such that $d(z_n, A\cap B(a,(1+\varepsilon)r_n)) \geq cr_n$. Of course $c \leq 1$ since $a \in A$, so $B(z_n, \varepsilon c r_n) \subset B(a,(1+\varepsilon)r_n) \setminus A$ and

$$\frac{\mu(B(a,(1+\varepsilon)r_n) \setminus A)}{\mu(B(a,(1+\varepsilon)r_n))} \geq \frac{\mu(B(z_n,\varepsilon c r_n))}{\mu(B(a,(1+\varepsilon)r_n))} \gtrsim 1$$

which contradicts that a is an m-density point of A.

■

If $A \subset Z$ and $a \in A$, any tangent space to $(A, d|_A)$ at a is naturally a subspace of a tangent space to (Z, d) at a. Equality holds if a is a t-density point of A.

Lemma 3.11 *Let a be a t-density point of a subset A of a doubling proper metric space Z. Suppose $r_n \downarrow 0$ is a decreasing sequence of radii such that the spaces $(Z_n, d_n = d/r_n, a)$ converge to a tangent (T, d, a). Let $A' \subset T$ be the set of points z such that there are $a_n \in Z_n \cap A$ such that $a_n \to z$. Then $A' = T$.*

Proof: Let $z \in T$, and consider a sequence of points z_n which tends to z. Since a is a t-density point, for each n we have

$$d(z_n, A) \leq \sup_{x\in B(a,|z_n-a|)} d(x, A\cap B(a, 2|z_n - a|))$$

so there is a point a_n in $A \cap B(a, 2|z_n - a|)$ such that

$$\lim_n \frac{|z_n - a_n|}{|z_n - a|} = 0\,.$$

Therefore,

$$d_n(z_n, a_n) \leq \frac{|z_n - a|}{r_n} \cdot \frac{|z_n - a_n|}{|z_n - a|}\,.$$

By definition, the first fraction is uniformly bounded and the second tends to 0: this proves that $z \in A'$.

■

3.6 Weak tangent spaces for topologically cxc maps

We now analyze weak tangents to metric spaces in the conformal gauge of a topologically cxc map f. Let d_v be a visual metric given by Theorem 1.5. The metric space (S^2, d_v) is doubling, a property preserved by quasisymmetries. Hence for any $d \in \mathcal{G}(f)$, any sequence $r_n \to 0$, and any sequence (x_n) of points in $X = (S^2, d)$, one may find a subsequence such that $(X, d/r_n, x_n)$ tends in the Gromov-Hausdorff topology to a doubling metric space (T, t).

Proposition 3.12 *For any $X \in \mathcal{G}(f)$, and any weak tangent space (T, t) of X, there is an open, onto map $h : T \to X$ with discrete fibers, and there is some constant $0 < c < 1$ such that, for any $R > 0$, there is some r_0 such that any ball $B(x, r)$ with $x \in B(t, R)$ and $r \in (0, r_0)$ contains a ball B of radius cr such that $g|_B$ is η-quasisymmetric.*

We will use the following lemma whose proof is left to the reader.

Lemma 3.13 *Let $X \in \mathcal{G}(f)$. Then any tangent space of X is quasisymmetrically equivalent to a tangent space of a visual metric metric (S^2, d_v).*

Proof: (Prop. 3.12) Lemma 3.13 implies that it is enough to treat the case $d = d_v$.

By assumption (T, t) is a limit of (X_n, x_n) where $X_n = (X, d_v/r_n)$ and (r_n) tends to zero.

Axiom [Irred] guarantees an integer n_0 so $f^{n_0}(U) = X$ for all $U \in \mathcal{U}_0$. By Proposition 3.8(3), there exists $K > 1$ such that for each n sufficiently large, there exists $W_n \in \mathbf{U}$ containing x_n such that $\mathrm{Round}(W_n, x_n) \leq K$ and $L(W_n, x_n) \asymp r_n$; that is, W_n is K-almost round about x_n and has diameter comparable to 1 in the rescaled metric d_v/r_n. Let $k_n = |W_n|$. ¿From the diameter estimates (3) of Theorem 1.5, the family of maps

$$(f^{k_n + n_0} : X_n \to X)_n$$

is uniformly Lipschitz for some constant independent of n. Passing to a subsequence, we may assume the sequence $(f^{k_n+n_0} : X_n \to X)_n$ converges (in the Gromov-Hausdorff sense) to a Lipschitz map $h : T \to X$. Since $f^{k_n+n_0}(W_n) = X$ by construction, the limit h is also onto.

Let us prove that h is open. Pick $z \in T$ and $r \in (0, \mathrm{diam} X)$, and choose $z_n \in X_n$ with $z_n \to z$. By the diameter estimates (3) of Theorem 1.5, $f^{k_n+n_0}(B_{X_n}(z_n, r))$ is ball centered at $f^{k_n+n_0}(z_n)$ with definite radius, at least some uniform constant $r' > 0$. It follows that, for n large enough, it covers $\overline{B(h(z), r'/2)}$, so $h(B(z, r)) \supset B(h(z), r'/2)$.

Similarly, Corollary 3.9 implies the quasisymmetry property; we leave the details to the reader.

Lastly, we prove that h has discrete fibers. Fix a large ball $B(t, R) \subset T$. Pick $W_n^R \in \mathbf{U}$ containing x_n such that $\mathrm{Round}(W_n^R, x_n) \le K$, $B(x_n, R \cdot r_n) \subset W_n^R$ and $L(W_n, x_n) \asymp R \cdot r_n$. For each n, $||W_n^R| - |W_n|| = O(\log R)$ holds so that $f^{k_n+n_0}|_{W_n^R}$ have multiplicity bounded by a constant depending only on R. This implies that $h|_{B(t,R)}$ has also bounded multiplicity.

■

Applying a theorem of Whyburn [Why, Thm. X.5.1], Theorem 3.2 implies

Corollary 3.14 *If $f : S^2 \to S^2$ is topologically cxc, and $X \in \mathcal{G}(f)$, then the map $h : T \to X$ given by Proposition 3.12 is a branched covering: it is locally of the form $z \mapsto z^k$ for some integer $k \ge 1$ and the branch set is discrete.*

4 Moduli of curves

This section discusses curves and different notions of moduli of family of curves.

4.1 Rectifiable curves and curve families

Let X be a proper metric space. In this subsection, we discuss some generalities on curves, see e.g [Väi, Chap. 1] for details.

A *curve* is a continuous map $\gamma : I \to X$ of a possibly infinite interval I into X; it is *degenerate* if it is constant, and *nondegenerate* otherwise. Given two curves $\gamma : I \to X$ and $\gamma' : I' \to X$, we say they *differ by a reparametrization* if there exists a (not necessarily strict) monotone and onto map $\alpha : I \to I'$ such that $\gamma = \gamma' \circ \alpha$ or $\alpha : I' \to I$ such that $\gamma' = \gamma \circ \alpha$. Note that two curves which differ by a reparametrization are simultaneously rectifiable or not.

We want to rescale curves of possibly variable length; for this, it is useful to think of such curves as maps of a common domain $\mathbb{R}$ into X. Let $\gamma : [0, \ell(\gamma)] \to X$ be a rectifiable curve parametrized by arclength. Given $t_0 \in [0, \ell(\gamma)]$ and $x_0 = \gamma(t_0)$, we define its *complete parametrization at* t_0, denoted (γ^0, t_0) as the map $\gamma^0 : (\mathbb{R}, 0) \to (X, x_0)$ defined by $\gamma^0(t) = \gamma(t+t_0)$ on $[-t_0, \ell(\gamma) - t_0]$, $\gamma_0|_{(-\infty,-t_0]} = \gamma(0)$ and $\gamma^0|_{[\ell(\gamma)-t_0,\infty)} = \gamma(\ell(\gamma))$. We let I_{γ^0} be the minimal compact interval such that $\gamma^0(I_{\gamma^0}) = \gamma^0(\mathbb{R})$. Note that given $x \in \gamma(I)$, there might be several complete parametrizations γ^0 such that $\gamma^0(0) = x$.

A sequence $(\gamma_n^0 : (\mathbb{R}, 0) \to (X, x_n))_n$ of completely parametrized curves *converges* if it converges uniformly on compact subsets of $\mathbb{R}$; note that this implies that the sequence x_n converges. The limit of such a sequence is a locally rectifiable curve. Given a bounded subset A of X, the set of completely parametrized curves $(\mathbb{R}, 0) \to (X, x)$ for which $x \in A$ is relatively compact.

Given a completely parametrized curve $\gamma : (\mathbb{R}, 0) \to (X, x_0)$ and a small constant $r > 0$, the rescaled curve given by $t \mapsto \gamma(t/r)$ defines again a completely parametrized curve $\gamma/r : (\mathbb{R}, 0) \to ((1/r)X, x_0)$. Note that the intervals on which these rescaled curves are nonconstant satisfy $I_{\gamma/r} = (1/r)I_{\gamma^0}$ and therefore grow to $\mathbb{R}$ as $r \to 0$.

The following proposition comes from [MgM, Lma 9.1].

Proposition 4.1 *Let $\gamma : I \to X$ be a rectifiable curve parametrised by arc length in a metric space X. Then*

$$\lim_{h\to 0} \frac{d(\gamma(t+h), \gamma(t-h))}{2|h|} = 1$$

for almost all $t \in I$.

Note that if

$$\lim_{h\to 0} \frac{d(\gamma(t+h), \gamma(t-h))}{2|h|} = 1$$

holds for some $t \in I$ then we also have

$$\lim_{h\to 0} \frac{d(\gamma(t+h), \gamma(t))}{|h|} = 1$$

since

$$\frac{d(\gamma(t+h), \gamma(t-h))}{2|h|} \leq \frac{1}{2}\left(\frac{d(\gamma(t+h), \gamma(t))}{|h|} + \frac{d(\gamma(t-h), \gamma(t))}{|h|}\right) \leq 1$$

and each term of the sum is also bounded by 1. That is, rectifiable curves are asymptotically geodesic near almost every point.

The next proposition illustrates how this property of rectifiable curves will be used to produce many geodesics on tangent spaces. The hypothesis gives some uniformity to the rate at which rectifiable curves become asymptotically geodesic.

Proposition 4.2 *Let (X, d) be a doubling proper metric space and (ε_k) be a sequence of positive reals tending to 0. Suppose $A \subset X$ and for each $x \in A$, there exists a completely parametrized curve $\gamma : (\mathbb{R}, 0) \to (X, x)$ such that, for all k,*

$$\sup_{|h|\leq 1/k} \left| \frac{d(\gamma(h), \gamma(-h))}{2|h|} - 1 \right| \leq \varepsilon_k \, .$$

Assume that x_0 is a point of t-density of A. Let (r_n) be a sequence of positive reals tending to zero, and assume that the limit of the rescaled spaces $X_n = (X, x_0, (1/r_n)d)$ exists in the pointed Gromov-Hausdorff topology. Let (Z, z_0) be the limit.

Then any point in Z lies in the image of a bi-infinite geodesic curve $\gamma : \mathbb{R} \to Z$ which is a limit of rescaled completely parametrized curves passing through points of A.

We say that the geodesic curve γ was *obtained by blowing up*. The rescaled curves will be completely parametrized curves in X_n which are rescalings of completely parametrized curves in X.

Proof: Let $z \in Z$; by Lemma 3.11, since x_0 is a t-density point, one may find a sequence of points $x_n \in X_n \cap A$ which tends to z. Consider complete parametrized curves $(\hat{\gamma}_n, x_n)$ in X. Let $\gamma_n(t) = \hat{\gamma}_n(r_n t)$ be its complete parametrization in X_n. Since the original parametrizations are by arc length, each γ_n is a rectifiable curve which is represented by a 1-Lipschitz function: let $t, t' \in \mathbb{R}$,

$$(1/r_n)d(\gamma_n(t), \gamma_n(t')) = (1/r_n)d(\hat{\gamma}_n(tr_n), \hat{\gamma}_n(t'r_n)) \le (1/r_n)|(tr_n) - (t'r_n)|\,.$$

Thus, Arzela-Ascoli's theorem implies the existence of a 1-Lipschitz limit $\gamma : \mathbb{R} \to Z$, with $\gamma(0) = z$.

We shall prove that γ is a geodesic.

By the choice of γ_n we have

$$\lim_{h\to 0} \sup_n \left| \frac{d(\hat{\gamma}_n(h), \hat{\gamma}_n(0))}{|h|} - 1 \right| = 0\,,$$

and

$$\lim_{n\to\infty} (1/r_n) d(\gamma_n(t), \gamma_n(0)) = \lim_{n\to\infty} |t| \frac{d(\hat{\gamma}_n(tr_n), \hat{\gamma}_n(0))}{|t| r_n} = |t|\,.$$

Hence $d(\gamma(t), \gamma(0)) = |t|$. Similarly,

$$\lim_{n\to\infty} (1/r_n) d(\gamma_n(t), \gamma_n(-t)) = \lim_{n\to\infty} |t| \frac{d(\hat{\gamma}_n(tr_n), \hat{\gamma}_n(-tr_n))}{|t| r_n} = 2|t|$$

and $d(\gamma(t), \gamma(-t)) = 2|t|$.

Let t, t' be non-zero real numbers. We may assume that $|t| \ge |t'|$.

We first assume that they have the same sign. Hence $|t - t'| = |t| - |t'|$. Therefore,

$$|t - t'| = |d(\gamma(t), \gamma(0)) - d(\gamma(t'), \gamma(0))| \le d(\gamma(t), \gamma(t')) \le |t - t'|\,.$$

If t and t' have opposite signs, then $|t - t'| = |t| + |t'|$ and $|t + t'| = |t| - |t'|$. Thus,

$$|t - t'| = 2|t| - (|t| - |t'|) = d(\gamma(t), \gamma(-t)) - d(\gamma(t'), \gamma(-t)) \le d(\gamma(t), \gamma(t')) \le |t - t'|\,.$$

Therefore, γ is geodesic.

∎

4.2 Analytic moduli

Suppose (X, d, μ) is a metric measure space, Γ is a family of curves in X, and $p \geq 1$. The *(analytic) p-modulus* of Γ is defined by

$$\mathrm{mod}_p\Gamma = \inf \int_X \rho^p d\mu$$

where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ such that ρ is *admissible*, i.e.

$$\int_\gamma \rho \, ds \geq 1$$

for all $\gamma \in \Gamma$ which are rectifiable. If Γ contains no rectifiable curves, $\mathrm{mod}_p\Gamma$ is defined to be zero. Note that when Γ contains a constant curve, then there are no admissible ρ, so we set $\mathrm{mod}_p\Gamma = +\infty$. We say that a family of curves is *nondegenerate* if it contains no constant curves. Modulus behaves like an outer measure: it is countably subadditive, etc.—see [HK].

We note that the moduli of two families of curves Γ and Γ' are the same if each curve of Γ' differs from a curve of Γ by reparametrization, and vice-versa.

We recall the following basic estimate in an Ahlfors-regular metric space [HK, Lma 3.14]:

Proposition 4.3 *Let X be a Q-Ahlfors regular metric space with $Q > 1$, Γ be the family of curves which joins $B(x, r)$ to $X \setminus B(x, R)$ for some $x \in X$, and $0 < r < R \leq \mathrm{diam}\, X$. Then*

$$\mathrm{mod}_Q\Gamma \lesssim \frac{1}{\log^{Q-1}(R/r)} .$$

In particular, the family of non-trivial curves which go through x has zero Q-modulus.

4.3 Thick curves

Suppose X is a Q-Ahlfors regular metric space. Denote by $\mathcal{P}(X)$ the set of compact continuous curves $\gamma : [0, 1] \to X$, endowed with the supremum-norm topology inherited from the metric of X. Thus, if γ is a curve, then $B_\infty(\gamma, r)$ denotes the set of curves $\gamma' : [0, 1] \to X$ such that $|\gamma(t) - \gamma'(t)| < r$ for all $t \in [0, 1]$.

Following Bonk and Kleiner, we say a curve $\gamma \in \mathcal{P}(X)$ is *thick* if it is nonconstant and the Q-modulus of $B_\infty(\gamma, \epsilon)$ is positive for all $\epsilon > 0$ [BnK3]. The following properties are established by Bonk and Kleiner:

Proposition 4.4 *Let X be an Ahlfors regular space. The set of non-thick curves has zero modulus. A nonconstant limit in $\mathcal{P}(X)$ of thick curves is thick, and nonconstant connected subcurves of thick curves are thick.*

If there exist curve families of positive modulus, the preceding proposition implies the existence of many thick curves. The next proposition refines and localizes this result.

Proposition 4.5 *Let X be an Ahlfors regular space and $\Gamma \subset \mathcal{P}(X)$ a family of nondegenerate curves of positive modulus. The subfamily Γ' of curves $\gamma \in \Gamma$ such that there is some $\varepsilon > 0$ with the property that $\mathrm{mod}_Q(B_\infty(\gamma, \varepsilon) \cap \Gamma) = 0$ has zero modulus.*

Proof: We may cover $\Gamma' \subset \mathcal{P}(X)$ by balls B_i of uniformly bounded radius such that $\mathrm{mod}_Q B_i \cap \Gamma = 0$. Since $\mathcal{P}(X)$ is separable, we may extract a countable subcover: the proposition follows from the σ-subadditivity of modulus.

■

Let us observe the following fact: let $\gamma : I \to X$ and $\gamma' : I' \to X$ be two curves which differ by a reparametrization. Given $\varepsilon > 0$, denote by Γ_ε (resp. Γ'_ε) the set of curves $c : I \to X$ (resp. $c' : I' \to X$) such that $\sup_{t\in I} |c(t) - \gamma(t)| \le \varepsilon$ (resp. $\sup_{t\in I'} |c'(t) - \gamma'(t)| \le \varepsilon$). Let $\alpha : I \to I'$ be a monotone onto map such that $\gamma = \gamma' \circ \alpha$. If $c' \in \Gamma'_\varepsilon$, then $(c' \circ \alpha) \in \Gamma_\varepsilon$. Conversely, assume that $c \in \Gamma_\varepsilon$. Since α is monotone, it is a uniform limit of homeomorphisms $\alpha_n : I \to I'$. Then, for all $t \in I'$,

$$|\alpha \circ \alpha_n^{-1}(t) - t| = |\alpha \circ \alpha_n^{-1}(t) - \alpha_n \circ \alpha_n^{-1}(t)| \le \|\alpha - \alpha_n\|_\infty .$$

Let ω be a modulus of continuity for γ'. It follows that

$$\begin{aligned} |c \circ \alpha_n^{-1}(t) - \gamma'(t)| &\le |c \circ \alpha_n^{-1}(t) - \gamma \circ \alpha_n^{-1}(t)| + |\gamma \circ \alpha_n^{-1}(t) - \gamma'(t)| \\ &\le \|c - \gamma\|_\infty + |\gamma'(\alpha \circ \alpha_n^{-1}(t)) - \gamma'(t)| \le \varepsilon + \omega(\|\alpha - \alpha_n\|_\infty) . \end{aligned}$$

Therefore, if n is large enough then $c \circ \alpha_n^{-1} \in \Gamma'_{2\varepsilon}$. ¿From this discussion, we see that thickness is a notion which may be generalized to curves defined on other intervals than $[0, 1]$, and that two curves which differ by a reparametrization are simultaneously thick or not.

We record here the following result:

Proposition 4.6 (Tyson) *Let X and Y be two regular metric spaces of dimension $Q > 1$ and $h : X \to Y$ be a quasisymmetric map. There exists a constant $K \ge 1$ such that, for any family of curves $\Gamma \subset X$,*

$$\frac{1}{K}\mathrm{mod}_Q\Gamma \le \mathrm{mod}_Q h(\Gamma) \le K\mathrm{mod}_Q\Gamma .$$

In particular, quasisymmetric maps preserve the set of thick curves.

For a proof, see [Tys, Thm. 1.4] (see also Prop. 5.2).

Note that thick curves need not exist: if there is no family of curves of positive modulus, which happens for instance if there are no rectifiable curves, then there are no thick curves.

4.4 Support of curve families

If Γ is family of curves, we define its support $\operatorname{supp}\Gamma$ as the set of points $x \in X$ such that $x \in \gamma$ for some $\gamma \in \Gamma$. We note that $\operatorname{supp}\Gamma$ might not be measurable. In any case, $\operatorname{supp}\Gamma$ is contained in some Borel set of the same measure. We will then denote by $\operatorname{supp}^*\Gamma$ such a set.

For $L > 0$, let $\Gamma_L = \{\gamma \in \Gamma,\ \ell(\gamma) \leq L\}$ be the subfamily of curves in Γ of length at most L; note that every element of Γ_L is rectifiable. We say that Γ is *closed* if, for all $L > 0$, the set of all complete parametrizations of elements of Γ_L is closed in the supremum norm on functions $\mathbb{R} \to X$. That is, whenever a sequence $(\gamma_n) \subset \Gamma_L$ has the property that some choice of complete parametrizations (γ_n^0) converges, then the limit is, up to reparametrization, a curve in Γ_L. Note that according to this definition the property of being closed depends only on the subfamily of rectifiable curves.

Lemma 4.7 *Let X be a Q-regular metric space for which there exists a family of non-degenerate curves of positive Q-modulus. Then there exists $L > 1$ and a closed family Γ of rectifiable curves of diameter at least $1/L$ and of length at most L and with compact support and positive modulus.*

Proof: According to Proposition 4.4, X contains a rectifiable thick curve $\gamma \in \mathcal{P}(X)$. Choose r so that $0 < r < \operatorname{diam}\gamma/3$ and put $\Gamma_0 = B_\infty(\gamma, r) \subset \mathcal{P}(X)$. The family Γ_0 has no trivial curves and each curve has diameter at least $\operatorname{diam}\gamma/3$. Define Γ_n as the curves of Γ_0 of length at most n. By the σ-subadditivity of moduli, there is some n such that $\operatorname{mod}_Q\Gamma_n > 0$. The closure Γ of the set of all complete parametrizations of elements of Γ_n in the sup-norm gives the desired family; the additional curves added as limit points are rectifiable.

■

4.5 Combinatorial moduli

After recalling the definition of combinatorial moduli, we establish and recall some basic estimates which will be used later on.

4.5.1 Definitions and properties

Definitions. Let $\mathcal{S}$ be a covering of a topological space X, and let $p \geq 1$. Denote by $\mathcal{M}_p(\mathcal{S})$ the set of functions $\rho : \mathcal{S} \to \mathbb{R}_+$ such that $0 < \sum \rho(s)^p < \infty$; elements of $\mathcal{M}_p(\mathcal{S})$ we call *admissible metrics*. For $K \subset X$ we denote by $\mathcal{S}(K)$ the set of elements of $\mathcal{S}$ which intersect K. The *ρ-length* of K is by definition

$$\ell_\rho(K) = \sum_{s \in \mathcal{S}(K)} \rho(s)\,.$$

Define the ρ-*volume* by

$$V_p(\rho) = \sum_{s \in \mathcal{S}} \rho(s)^p .$$

If Γ is a family of curves in X and if $\rho \in \mathcal{M}_p(\mathcal{S})$, we define

$$L_\rho(\Gamma, \mathcal{S}) = \inf_{\gamma \in \Gamma} \ell_\rho(\gamma),$$

$$\mathrm{mod}_Q(\Gamma, \rho, \mathcal{S}) = \frac{V_p(\rho)}{L_\rho(\Gamma, \mathcal{S})^p},$$

and the *combinatorial modulus* by

$$\mathrm{mod}_p(\Gamma, \mathcal{S}) = \inf_{\rho \in \mathcal{M}_p(\mathcal{S})} \mathrm{mod}_p(\Gamma, \rho, \mathcal{S}).$$

Note that if $\mathcal{S}$ is a finite cover, then the modulus of a nonempty family of curves is always finite and positive.

A metric ρ for which $\mathrm{mod}_p(\Gamma, \rho, \mathcal{S}) = \mathrm{mod}_p(\Gamma, \mathcal{S})$ will be called *optimal.* We will consider here only finite covers; in this case the proof of the existence of optimal metrics is a straightforward argument in linear algebra. The following result is the analog of the classical Beurling's criterion which characterises optimal metrics.

Proposition 4.8 *Let $\mathcal{S}$ be a finite cover of a space X, Γ a family of curves and $p > 1$. An admissible metric ρ is optimal if and only if there is a non-empty finite subfamily $\Gamma_0 \subset \Gamma$ and non-negative scalars λ_γ, $\gamma \in \Gamma_0$, such that*

1. *for all $\gamma \in \Gamma_0$, $\ell_\rho(\gamma) = L_\rho(\Gamma, \mathcal{S})$;*
2. *for any $s \in \mathcal{S}$,*
$$p\rho(s)^{p-1} = \sum \lambda_\gamma$$
where the sum is taken over curves in Γ_0 which go through s.

Moreover, an optimal metric is unique up to scale, and one has

$$\mathrm{mod}_p(\Gamma, \mathcal{S}) = \frac{1}{p} \sum_{\gamma \in \Gamma_0} \lambda_\gamma .$$

For a proof, see Proposition 2.1 and Lemma 2.2 in [Haï].

Proposition 4.9 *Let $\mathcal{S}$ be a locally finite cover of a topological space X and $p \geq 1$.*

1. *If $\Gamma_1 \subset \Gamma_2$ then $\mathrm{mod}_p(\Gamma_1, \mathcal{S}) \leq \mathrm{mod}_p(\Gamma_2, \mathcal{S})$.*
2. *If every curve of Γ_1 contains a curve of Γ_2 then $\mathrm{mod}_p(\Gamma_1, \mathcal{S}) \leq \mathrm{mod}_p(\Gamma_2, \mathcal{S})$.*

3. Let $\Gamma_1, \ldots, \Gamma_n$ be a set of curve families in X. Then

$$\mathrm{mod}_p(\cup\Gamma_j, \mathcal{S}) \leq \sum \mathrm{mod}_p(\Gamma_j, \mathcal{S}) \,.$$

The proof is the same as the standard one for classical moduli (see for instance [Väi, Thm. 6.2, Thm. 6.7]) and so is omitted.

4.5.2 Dimension comparison

Let X be a topological space endowed with a finite cover $\mathcal{S}$. Consider a curve family Γ. For $p \geq 1$, when considering admissible metrics ρ for the p-modulus, one may always assume that $L_\rho(\Gamma, \mathcal{S}) = 1$ and $\rho \leq 1$. Thus, if $1 \leq p \leq q$, then $\mathrm{mod}_q(\Gamma, \mathcal{S}) \leq \mathrm{mod}_p(\Gamma, \mathcal{S})$. The following proposition improves in some cases this estimate:

Proposition 4.10 *If $1 \leq p \leq q$, then, for all $\varepsilon > 0$, the following holds:*

$$\mathrm{mod}_q(\Gamma, \mathcal{S}) \leq \left(\varepsilon^{q-p} + \frac{1}{\varepsilon^p} \sup_{s\in\mathcal{S}} \mathrm{mod}_q(\Gamma(s), \mathcal{S})\right) \mathrm{mod}_p(\Gamma, \mathcal{S})$$

where $\Gamma(s)$ denotes the subfamily of curves of Γ which go through s.

Proof: Let ρ be the optimal metric for the p-modulus of Γ such that $L_\rho(\Gamma, \mathcal{S}) = 1$. For $\varepsilon > 0$, set $\mathcal{E}(\varepsilon) = \{s \in \mathcal{S},\ \rho(s) \geq \varepsilon\}$. It follows from the Markov inequality that

$$card\, \mathcal{E}(\varepsilon) \leq \frac{1}{\varepsilon^p} \mathrm{mod}_p(\Gamma, \mathcal{S}) \,.$$

Decompose Γ into two family of curves: those which go through at least one element of $\mathcal{E}(\varepsilon)$, which we will denote by $\Gamma_{\mathcal{E}}$, and its complement which we denote by Γ'. Then, by the elementary properties of moduli,

$$\begin{cases} \mathrm{mod}_q(\Gamma', \mathcal{S}) \leq \displaystyle\sum_{s\in\mathcal{S}(\Gamma')} \rho^q(s) \leq \left(\sup_{\mathcal{S}\setminus\mathcal{E}(\varepsilon)} \rho\right)^{q-p} \sum_{s\in\mathcal{S}(\Gamma')} \rho^p(s) \leq \varepsilon^{q-p} \mathrm{mod}_p(\Gamma, \mathcal{S}) \,, \\ \mathrm{mod}_q(\Gamma_{\mathcal{E}}, \mathcal{S}) \leq \displaystyle\sum_{s\in\mathcal{E}(\varepsilon)} \mathrm{mod}_q(\Gamma(s), \mathcal{S}) \leq \left(\frac{1}{\varepsilon^p} \mathrm{mod}_p(\Gamma, \mathcal{S})\right) \sup_{s\in\mathcal{S}} \mathrm{mod}_q(\Gamma(s), \mathcal{S}) \,. \end{cases}$$

Hence

$$\mathrm{mod}_q(\Gamma, \mathcal{S}) \leq \left(\varepsilon^{q-p} + \frac{1}{\varepsilon^p} \sup_{s\in\mathcal{S}} \mathrm{mod}_q(\Gamma(s), \mathcal{S})\right) \mathrm{mod}_p(\Gamma, \mathcal{S}) \,.$$

■

Corollary 4.11 *Let $(\mathcal{S}_n)$ be a sequence of finite coverings and $q > p \geq 1$. If we assume there is a sequence (η_n) of positive numbers tending to zero such that*

$$\sup_{s \in \mathcal{S}_n} \mathrm{mod}_q(\Gamma(s), \mathcal{S}_n) \leq \eta_n \,,$$

then

$$\lim_{n\to\infty} \frac{\mathrm{mod}_q(\Gamma, \mathcal{S}_n)}{\mathrm{mod}_p(\Gamma, \mathcal{S}_n)} = 0 \,.$$

This is yet further evidence that combinatorial modulus behaves like Hausdorff measure, under the appropriate assumptions.

Proof: Pick $\varepsilon_n = \eta_n^{1/(2p)}$ so that (ε_n) converges to 0, and apply Proposition 4.10 for each n. It follows that

$$\frac{\mathrm{mod}_q(\Gamma, \mathcal{S}_n)}{\mathrm{mod}_p(\Gamma, \mathcal{S}_n)} \leq \varepsilon_n^{q-p} + \frac{1}{\varepsilon_n^p}\eta_n \leq \varepsilon_n^{q-p} + \eta_n^{1/2} \,.$$

Therefore, since $q > p$,

$$\lim_{n\to\infty} \frac{\mathrm{mod}_q(\Gamma, \mathcal{S}_n)}{\mathrm{mod}_p(\Gamma, \mathcal{S}_n)} = 0 \,.$$

■

4.5.3 Transformation rules

Proposition 4.12 *Let X and X' be two connected, Hausdorff and locally compact topological spaces, and $\mathcal{S}$, $\mathcal{S}'$ respectively be coverings by compact connected subsets. Let $f : X' \to X$ be an onto, proper and continuous map such that*

- *for every $s' \in \mathcal{S}'$, $f(s') \in \mathcal{S}$ and for every $s \in \mathcal{S}$, the set of connected components of $f^{-1}(\{s\})$ is a subset of $\mathcal{S}'$;*
- *there exists an integer $d \geq 1$ such that, for every $s \in \mathcal{S}$, the set $f^{-1}(\{s\})$ has at most d connected components.*

Let $\Gamma' \subset X'$ and $\Gamma \subset X$ be two family of curves. Then

1. *if, for every $\gamma' \in \Gamma'$, $f(\gamma')$ contains a curve $\gamma \in \Gamma$, then*

$$\mathrm{mod}_p(\Gamma', \mathcal{S}') \leq d \cdot \mathrm{mod}_p(\Gamma, \mathcal{S}) \,;$$

2. *if every curve in Γ contains the image of a curve of Γ', then*

$$\mathrm{mod}_p(\Gamma', \mathcal{S}') \geq \frac{1}{d^p}\mathrm{mod}_p(\Gamma, \mathcal{S}) \,.$$

Proof:

1. Let ρ be an admissible metric for Γ. Set $\rho' = \rho \circ f$. If $\gamma' \in \Gamma'$, then let $\gamma \in \Gamma$ be a subcurve of $f(\gamma')$. One has $f(\mathcal{S}'(\gamma')) = \mathcal{S}(f(\gamma'))$ so

$$\ell_{\rho'}(\gamma') \geq \sum_{s \in \mathcal{S}(f(\gamma'))} \rho(s) \geq \sum_{s \in \mathcal{S}(\gamma)} \rho(s) \geq L(\Gamma, \rho)\,.$$

On the other hand, $V_p(\rho') \leq d \cdot V_p(\rho)$ so that

$$\mathrm{mod}_p(\Gamma', \mathcal{S}') \leq d\,\mathrm{mod}_p(\Gamma, \mathcal{S})\,.$$

2. Let ρ' be an admissible metric for Γ'. Set

$$\rho(s) = \left(\sum_{f(s')=s} \rho'(s')^p\right)^{1/p}\,.$$

It follows that

$$\rho(s) \geq \max_{f(s')=s} \rho'(s') \geq \frac{1}{d} \sum_{f(s')=s} \rho'(s')\,.$$

Therefore, if $\gamma \in \Gamma$ and γ' is a curve the image of which is contained in γ, then, as $\mathcal{S}'(\gamma') \subset \mathcal{S}'(f^{-1}(\gamma))$,

$$\ell_\rho(\gamma) \geq \frac{1}{d}\ell_{\rho'}(\gamma') \geq \frac{1}{d}L(\Gamma', \rho')\,.$$

Moreover, $V_p(\rho) = V_p(\rho')$ so that

$$\mathrm{mod}_p(\Gamma', \mathcal{S}') \geq \frac{1}{d^p}\mathrm{mod}_p(\Gamma, \mathcal{S})\,.$$

■

4.5.4 Analytic versus combinatorial moduli

Under suitable conditions, the combinatorial moduli obtained from a sequence $(\mathcal{S}_n)$ of coverings can be used to approximate analytic moduli on metric measure spaces.

The approximation result we use requires the sequence of coverings $(\mathcal{S}_n)$ to be a *uniform family of quasipackings*.

Definition 4.13 (Quasipacking) *A* quasipacking *of a metric space is a locally finite cover $\mathcal{S}$ such that there is some constant $K \geq 1$ which satisfies the following property. For any $s \in \mathcal{S}$, there are two balls $B(x_s, r_s) \subset s \subset B(x_s, K \cdot r_s)$ such that the family $\{B(x_s, r_s)\}_{s \in \mathcal{S}}$ consists of pairwise disjoint balls. A family $(\mathcal{S}_n)$ of quasipackings is called* uniform *if the mesh of $\mathcal{S}_n$ tends to zero as $n \to \infty$ and the constant K defined above can be chosen independent of n.*

Uniform quasipackings are preserved under quasisymmetric maps quantitatively.

The next result says that under appropriate hypotheses, analytic and combinatorial moduli are comparable.

Proposition 4.14 *Suppose $Q > 1$, X is an Ahlfors Q-regular compact metric space, and $(\mathcal{S}_n)$ is a sequence of uniform quasipackings. Let Γ be a nondegenerate closed family of curves in X. Then either*

1. $\mathrm{mod}_Q \Gamma = 0$ *and* $\lim_{n\to\infty} \mathrm{mod}_Q(\Gamma, \mathcal{S}_n) = 0$*, or*
2. $\mathrm{mod}_Q \Gamma > 0$*, and there exist constants $C \geq 1$ and $N \in \mathbb{N}$ such that for any $n > N$,*

$$\frac{1}{C}\mathrm{mod}_Q(\Gamma, \mathcal{S}_n) \leq \mathrm{mod}_Q \Gamma \leq C \mathrm{mod}_Q(\Gamma, \mathcal{S}_n).$$

See Proposition B.2 in [Haï].

Suppose now $f : S^2 \to S^2$ is topologically cxc, let $\mathcal{S}_n := \mathcal{U}_n, n = 0, 1, 2, \ldots$ be the sequence of open covers as in the definition, and let d be a metric in the conformal gauge of f. Then, according to [HP2, Thm. 4.1], $\{\mathcal{S}_n\}$ is a uniform sequence of quasipackings, so we may apply Proposition 4.14 to estimate analytic moduli using combinatorial moduli.

4.6 Intersection of curves

Proposition 4.15 *Let X be a Q-Ahlfors regular compact metric space, $Q > 2$. If $\Gamma \subset \mathcal{P}(X)$ is a closed nondegenerate curve family of positive Q-modulus, then the family*

$$\Gamma^{\perp} = \{\gamma' \in \mathcal{P}(X) \mid \forall \gamma \in \Gamma,\ \gamma' \cap \gamma \neq \emptyset\}$$

is also closed and nondegenerate, and $\mathrm{mod}_Q \Gamma^{\perp} = 0$.

Proof: That $\Gamma^{\perp}$ is closed follows immediately from the definitions. If $\Gamma^{\perp}$ contained a constant curve then every curve of Γ would go through a given point, implying $\mathrm{mod}_Q \Gamma = 0$ by Proposition 4.3; hence $\Gamma^{\perp}$ is nondegenerate.

Given $\delta > 0$, denote by $\Gamma^{\perp}_{\delta}$ the family of curves $c \in \Gamma^{\perp}$ of diameter at least δ. The σ-subadditivity of the modulus implies that it is enough to prove that $\mathrm{mod}_Q \Gamma^{\perp}_{\delta} = 0$ for all $\delta > 0$. Fix $\delta > 0$, and note that $\Gamma_\delta^\perp$ is closed, so we may estimate its modulus using quasipackings, cf. Proposition 4.14.

Let $\mathcal{R}_n$ denote a maximal 2^{-n}-separated set of X, then $\mathcal{S}_n = \{B(x, 1/2^n), x \in \mathcal{R}_n\}$ defines a uniform sequence of quasipackings. It follows from the Q-regularity of X that

$$\lim_{n\to\infty} \sup_{s\in\mathcal{S}_n} \mathrm{mod}_Q(\Gamma_\delta(s), \mathcal{S}_n) = 0\,,$$

where $\Gamma_\delta(s)$ denotes the set of curves of diameter at least δ which intersects s. This is just the discrete version of Proposition 4.3: given $s_0 = B(x_0, 1/2^n) \in \mathcal{S}_n$, one may estimate $\mathrm{mod}_Q(\Gamma_\delta(s_0), \mathcal{S}_n)$ by setting $\rho_n(s) = 0$ if $dist(x_0, s) > \delta/2$ or $s = s_0$, and $\rho_n(s) = \mathrm{diam}\, s/dist(x_0, s)$ otherwise.

Fix $n \geq 1$. By Proposition 4.8, there are an optimal metric ρ_n for $\mathrm{mod}_2(\Gamma_\delta^\perp, \mathcal{S}_n)$, a nonempty subfamily $\Gamma_n \subset \Gamma_\delta^\perp$ and nonnegative scalars λ_γ, $\gamma \in \Gamma_n$, such that

1. for all $\gamma \in \Gamma_n$, $\ell_{\rho_n}(\gamma) = L_{\rho_n}(\Gamma_\delta, \mathcal{S}_n) = 1$;
2. for any $s \in \mathcal{S}_n$,
$$2\rho_n(s) = \sum \lambda_\gamma$$
where the sum is taken over curves in Γ_n which go through s.

We note that any curve of Γ intersects each curve of Γ_n so that

$$L_{\rho_n}(\Gamma, \mathcal{S}_n) \geq \frac{1}{2} \sum_{\gamma\in\Gamma_n} \lambda_\gamma = v_2(\rho_n)\,.$$

It follows that

$$\mathrm{mod}_2(\Gamma, \mathcal{S}_n) \cdot \mathrm{mod}_2(\Gamma_\delta^\perp, \mathcal{S}_n) \leq \left(\frac{v_2(\rho_n)}{v_2(\rho_n)^2}\right) v_2(\rho_n) \leq 1\,.$$

According to Proposition 4.14, one has, for any n large enough,

$$0 < \mathrm{mod}_Q \Gamma \lesssim \mathrm{mod}_Q(\Gamma, \mathcal{S}_n) \leq \mathrm{mod}_2(\Gamma, \mathcal{S}_n)$$

since $Q \geq 2$. Therefore,

$$\mathrm{mod}_2(\Gamma_\delta^\perp, \mathcal{S}_n) \lesssim \frac{1}{\mathrm{mod}_Q \Gamma} < \infty$$

for all n. Since $Q > 2$, we may apply Corollary 4.11 to $\Gamma_\delta^\perp$ with $q = Q$ and $p = 2$ to conclude

$$\lim_{n\to\infty} \mathrm{mod}_Q(\Gamma_\delta^\perp, \mathcal{S}_n) = 0\,.$$

■

Let X be a metric space and suppose $\gamma, \gamma' \in \mathcal{P}(X)$ are two curves in X. We say that γ and γ' *cross* if there exists $\epsilon > 0$ such that each curve of $B_\infty(\gamma, \epsilon)$ meets each curve of $B_\infty(\gamma', \epsilon)$.

Corollary 4.16 *Let X be a Q-regular metric space with $Q \geq 2$. If there are two thick curves which cross, then $Q = 2$.*

Proof: Suppose γ and γ' are two thick curves. By definition, for all $\epsilon > 0$, we have $\text{mod}_Q B_\infty(\gamma, \epsilon) > 0$ and $\text{mod}_Q B_\infty(\gamma', \epsilon) > 0$. If γ, γ' cross, then for some ϵ we have $B_\infty(\gamma', \epsilon) \subset B_\infty(\gamma, \epsilon)^\perp$. But Proposition 4.15 implies $\text{mod}_Q B_\infty(\gamma, \epsilon)^\perp = 0$ and so $\text{mod}_Q B_\infty(\gamma', \epsilon) = 0$, which contradicts the thickness of γ'.

■

We may now give a necessary and sufficient condition for a dynamical system to be conjugate to a conformal one:

Proposition 4.17 *Let $\mathcal{D}$ be either a topologically cxc mapping on S^2 or a hyperbolic group with boundary S^2. Assume that the conformal dimension of $\mathcal{D}$ is attained by an Ahlfors Q-regular metric space X in its conformal gauge. Then $\mathcal{D}$ is topologically conjugate to a semihyperbolic rational map or cocompact Kleinian group if and only if there are two thick curves in X which cross.*

Proof: Let d be a metric of minimal dimension $Q \geq 2$.

If $\mathcal{D}$ is conjugate to a genuine conformal dynamical system, then $Q = 2$, and there is a quasisymmetric map from (X, d) to $\widehat{\mathbb{C}}$ by Theorem 1.5. Since the sphere admits thick curves which cross, so does X, according to Proposition 4.6.

Conversely, if there exist thick curves in X which cross, Corollary 4.16 implies that $Q = 2$. Theorem 1.6 completes the proof.

■

5 Moduli at the minimal dimension

In this section, we assume that $f : S^2 \to S^2$ is topologically cxc, and that d is a Q-dimensional Ahlfors regular metric on S^2 belonging to the conformal gauge of f. We denote by $X = (S^2, d)$. Recall that the sequence $\mathcal{S}_n := \mathcal{U}_n$ of coverings forms a uniform sequence of quasipackings.

5.1 Positive modulus on X

Proposition 5.1 *There is a family of curves on X with positive Q-modulus.*

Proof: Since Q is the AR-conformal dimension, it follows from [KL, Cor. 1.0.2] that a weak tangent (T,t) of X admits a family Γ of positive Q-modulus contained in some ball $B_T(t,R)$ for some $R>0$. We may assume that this family of curves is closed with compact support and definite diameter (Lemma 4.7). By Proposition 4.5, we may also assume that $\mathrm{mod}_Q(B_\infty(\gamma,\varepsilon)\cap\Gamma)>0$ for every curve γ in Γ and any $\varepsilon>0$. Let $c\in(0,1)$ be the constant given by Proposition 3.12.

Since Γ has positive modulus, its support $\mathrm{supp}\,\Gamma$ has positive measure, so we may find a point of m-density $z\in\mathrm{supp}\,\Gamma$. It follows that we may find a radius $r>0$ which is small enough so that any ball $B\subset B(z,r)$ of radius at least $cr/3$ satisfies $\mu(B\cap\mathrm{supp}\,\Gamma)>0$.

According to Proposition 3.12, there exists a map $h:T\to X$ and a ball $B\subset B(z,r)$ of radius cr such that $h:B\to X$ is a quasisymmetric embedding. Since $\mu(\mathrm{supp}\,\Gamma\cap(1/3)B)>0$ by construction, there is some curve $\gamma\in\Gamma$ which intersects $(1/3)B$ hence there is some $\varepsilon>0$ such that every curve from $\Gamma\cap B_\infty(\gamma,\varepsilon)$ intersects $(1/2)B$. It follows that $\mathrm{mod}_Q\Gamma_0>0$ where Γ_0 denotes the subcurves in B of those curves in Γ which enter $(1/2)B$.

Therefore, from Proposition 4.6, it follows that $h(\Gamma_0)$ is a family of positive Q-modulus on X.

■

5.2 Invariance of thick curves

Proposition 5.2 *The image of a thick curve under f is a thick curve.*

Proof: Let γ be a thick curve and let $\varepsilon>0$. Since f is uniformly continuous, there exists $\delta>0$ such that $f(B_\infty(\gamma,\delta))\subset B_\infty(f(\gamma),\varepsilon)$. By definition $\mathrm{mod}_Q B_\infty(\gamma,\delta)>0$; there is some $L<\infty$ such that the set $\Gamma\subset B_\infty(\gamma,\delta)$ of curves of length at most L has positive Q-modulus. According to Proposition 4.14, there exists $m>0$ such that

$$\mathrm{mod}_Q(\Gamma,\mathcal{S}_n)\geq m>0$$

for all n large enough. Therefore, by Proposition 4.12, $\mathrm{mod}_Q(f(\Gamma),\mathcal{S}_n)\gtrsim m$ holds, and another appeal to Proposition 4.14 now implies that

$$\mathrm{mod}_Q B_\infty(f(\gamma),\varepsilon)\geq\mathrm{mod}_Q f(\Gamma)>0\,.$$

■

The following corollary is needed for the construction of the second tangent space, T_2, mentioned in the introduction.

Corollary 5.3 *If T is a tangent space of X, $h : T \to X$ be given by Proposition 3.12, and $\gamma : [0,1] \to T$ is a nonconstant limit of thick curves in X, then $h \circ \gamma$ is thick.*

Proof: Let $X_n = (X_n, d/r_n)$ and suppose $(X_n, x_n) \to (T, t)$ is a tangent space. As in Proposition 3.12, suppose $f^{k_n+n_0} : (X, d/r_n, x_n) \to (X, d, y_n)$ tends to $h : (T,t) \to (X, d, y)$. The hypothesis implies that there exist thick curves $\gamma_n : [0,1] \to X_n$ converging to $\gamma : [0,1] \to T$. The definitions of convergence of maps imply that the curve $h \circ \gamma : [0,1] \to X$ is the uniform limit of the sequence of curves $\beta_n = f^{k_n+n_0} \circ \gamma_n$. By Proposition 5.2, each curve β_n is thick. Since nonconstant limits of thick curves are thick (Proposition 4.4), γ is thick.

■

6 Non-crossing curves

In this section, we prove the following:

Theorem 6.1 *Assume that $f : S^2 \to S^2$ is topologically cxc and $X \in \mathcal{G}(f)$ is a Q-regular 2-sphere with $Q = \mathrm{confdim}_{AR}\mathcal{G}(f) > 2$. Then f is topologically conjugate to a Lattès example.*

In the remainder of this section, f and X satisfy the hypotheses of Theorem 6.1. ¿From Propositions 5.1 and 4.4, we obtain a family Γ of thick curves of positive Q-modulus on X; by Lemma 4.7 we may assume it is closed and that its elements have diameters bounded above and below. Since $Q > 2$, Proposition 4.15 implies that thick curves on X cannot cross.

6.1 Foliation by thick curves

Proposition 6.2 *There exist a tangent space T of X and a foliation $\mathcal{F}_T$ of T by bi-infinite geodesics such that each leaf is a limit of rescaled thick curves of X.*

¿From this, we will deduce:

Corollary 6.3 *The space X admits a foliation $\mathcal{F}_X$ with finitely many singularities invariant by f. More precisely, there exists a finite set F such that $X \setminus F$ is foliated by locally thick curves, each point of F is a one-prong singularity of $\mathcal{F}_X$, and $f^{-1}(\mathcal{F}_X \setminus F) \subset \mathcal{F}_X \setminus F$.*

This proves Theorem 1.2 in the iterated case.

By a one-prong singularity of a foliation of a surface, we mean that the foliation near that point is equivalent to the singular foliation of the plane near the origin obtained by starting with the horizontal foliation on the complex plane and identifying points z and $-z$.

The proofs occupy the remainder of this subsection.

6.1.1 Thick crosscuts

Let D be a Jordan domain in X. A *crosscut* is a curve γ such that $\gamma(0), \gamma(1) \in \partial D$, $\gamma(0) \neq \gamma(1)$, and $\gamma(0,1) \subset D$. Let us say that a crosscut γ is thick if, for all $\varepsilon > 0$, $\mathrm{mod}_Q \Gamma_\varepsilon > 0$, where Γ_ε denotes the subset of crosscuts of $B_\infty(\gamma, \varepsilon)$.

The main results of this paragraph are

Proposition 6.4 *Let γ be a thick crosscut of D.*

1. *There are exactly two components $D_\pm$ of $D \setminus \gamma(0,1)$ for which the boundary intersects ∂D.*
2. *The image $\gamma[0,1]$ is either contained in ∂D_+ or in ∂D_-.*
3. *Any other component W is a Jordan domain, and there are parameters $0 < s < t < 1$ such that $\partial W \subset \gamma([s,t])$ with $\gamma(s) = \gamma(t)$.*

and

Proposition 6.5 *Let γ_0 and γ_+ and γ_- be three thick crosscuts of D with endpoints in the following cyclic order:*

$$\gamma_+(0) < \gamma_0(0) < \gamma_-(0) < \gamma_-(1) \leq \gamma_0(1) \leq \gamma_+(1) < \gamma_+(0)\,.$$

Then $\gamma_0 \cap \gamma_- \cap \gamma_+ \cap D = \emptyset$.

We first establish some preliminary facts. The definitions imply at once the following fact:

Fact 6.6 *Let γ, γ' be two crosscuts with endpoints in the following cyclic order*

$$\gamma(0) < \gamma'(0) < \gamma(1) < \gamma'(1)\,.$$

Then γ and γ' cannot be thick simultaneously.

Fact 6.7 *Let γ_0 be a thick crosscut, and let $\gamma_1 : [0,1] \to D$ be thick. Then γ_1 intersects at most one connected component of $D \setminus \gamma_0$.*

Proof: Let us proceed by contradiction and assume that $\gamma_1(0)$ and $\gamma_1(1)$ lie in different components U and V of $D \setminus \gamma_0$. By Fact 6.6, we may also assume that $\partial U \subset \gamma_0$.

Let $\tau = \sup\{t > 0,\ \gamma_1(t) \in U\}$ and $z_0 = \gamma_1(\tau)$. By construction, (a) $z_0 \in \partial U$; (b) for any $\varepsilon > 0$, there is some $t \in (\tau - \varepsilon, \tau)$ with $\gamma_1(t) \in U$.

Let $s \in (0,1)$ with $\gamma_1([s,1]) \subset V$. Let $D' \subset D$ be a Jordan neighborhood of $\gamma_1([\tau, s])$. If it is small enough, then $\gamma_1|_{[a,b]}$ is a crosscut of D', where (a,b) is the connected component of $\gamma_1^{-1}(D')$ containing τ, with $\gamma_1(a) \in U$ and $\gamma_1(b) \in V$. Furthermore, the connected component γ_0' of $\gamma_0 \cap D'$ which contains z_0 separates $\gamma_1(a)$ from $\gamma_1(b)$ otherwise we would have $U = V$.

Therefore, there are two intervals $[s_1, s_2]$ and $[t_1, t_2]$ such that $\gamma_0(s_1), \gamma_0(t_1) \in \partial D'$, $\gamma_0(s_2) = \gamma_0(t_2) = z_0$ and $\gamma_0((s_1, s_2]) \cup \gamma_0((t_1, t_2])$ is the image of a crosscut of D' which separates $\gamma_1|_{[a,b]}$. Since γ_1 is thick, one may find another thick curve γ_2 arbitrarily close to γ_1 which avoids z_0. We may thus extract a crosscut of D' from γ_2 which either intersects $\gamma_0((s_1, s_2))$ or $\gamma_0((t_1, t_2))$. In both cases, we may reduce D' and apply Fact 6.6 to obtain a contradiction.

■

We now turn to the proofs of the propositions.

Proof: (Prop. 6.4) The first point is clearly true since γ is a crosscut. Note that if γ is not contained in ∂D_+, then there is some parameter t such that $d(\gamma(t), D_+) \geq \delta > 0$. Therefore, any thick curve at distance at most $\delta/2$ from γ cannot lie in D_+ by Fact 6.7; similarly for D_-. But any crosscut has to intersect $D_+ \cup D_-$, so Fact 6.7 yields a contradiction. This proves 2.

Let us prove the last point: let W be a component of $D \setminus \gamma(0,1)$ different from $D_\pm$. Then ∂W is contained in $\gamma(0,1)$ which is locally connected, so that Carathéodory's theorem implies that any conformal map $h : \mathbb{D} \to W$ extends continuously to the boundary. If ∂W is not a Jordan domain, then there are two rays in $\mathbb{D}$ which are mapped to a Jordan curve in $\overline{W}$ which separates γ. Fact 6.7 yields another contradiction. So W is a Jordan domain. Let $s = \min \gamma^{-1}(\partial W)$ and $t = \max \gamma^{-1}(\partial W)$: by construction, $\partial W \subset \gamma([s,t])$. If $\gamma(s) \neq \gamma(t)$, then $\partial W \setminus \{\gamma(s), \gamma(t)\}$ are two arcs $a_\pm$. Note that since $c_\pm = \gamma[0,s] \cup a_\pm \cup \gamma[t,1]$ are crosscuts contained in γ and $a_+ \cap c_- = a_- \cap c_+ = \emptyset$, point 2. above cannot hold, so we obtain a contradiction and $\gamma(s) = \gamma(t)$.

■

Proof: (Prop. 6.5) It follows from Fact 6.7 that we may assume that γ_- and γ_+ lie in different components of $D \setminus \gamma_0$. Let us assume that they all meet at a point $z_0 \in D$.

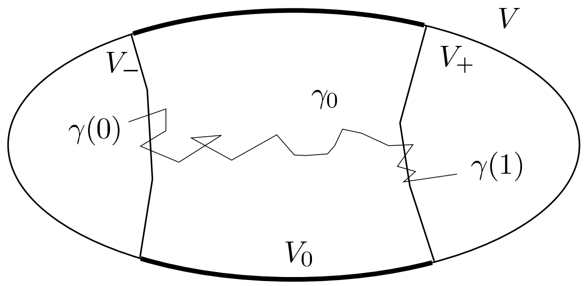
One may find some $\varepsilon > 0$ such that $\gamma_\pm \notin B_\infty(\gamma_0, \varepsilon)$. By Fact 6.7, any thick curve in $B_\infty(\gamma_0, \varepsilon)$ is squeezed between γ_- and γ_+: thus, they all go through z_0, which is impossible by Proposition 4.3 and the definition of thick curves.

■

6.1.2 Relation order on thick curves

Let $\gamma_0 : [0,1] \to X$ be a parametrized rectifiable thick curve of X. We will define an ordering on a space Γ of parametrized thick curves γ that are close to γ_0 in $\mathcal{P}(X)$. This ordering will be a reflexive and transitive binary relation (but might not be antisymmetric).

First, we choose γ_0 in a convenient way. Let V be a Jordan domain which contains γ_0 in its interior. Decompose V into three open Jordan subdomains V_0, V_- and V_+ as shown. That is, there is a homeomorphism of V to $[-2,2]\times[0,1]$ such that V_- is mapped to $[-2,-1]\times[0,1]$, V_+ is mapped to $[1,2]\times[0,1]$, $\gamma(0)\in V_-$ and $\gamma(1)\in V_+$.



Since γ_0 is rectifiable and compact, there are at most finitely many subcurves of γ_0 in V connecting V_- to V_+. Restricting to such a subcurve if necessary and reparametrizing, Proposition 4.4 implies that we may assume that there is exactly one such subcurve.

Fix $\gamma_0, V_0, V_\pm, V$ as in the preceding paragraph. Next, we define Γ, a set of suitable curves close to γ_0. There exists $r>0$ small enough so that for any curve $\gamma\in\overline{B_\infty(\gamma_0,r)}$, we have (i) $\gamma\subset V$, (ii) $\gamma(0)\in V_-, \gamma(1)\in V_+$, and (iii) there is exactly one connected component of $\gamma\cap V_0$ whose closure intersects both ∂V_+ and ∂V_-; we denote this distinguished component by γ_d. By Proposition 4.7, there exists a compact nondegenerate subfamily $\Gamma\subset\overline{B_\infty(\gamma_0,r)}$ comprised of rectifiable thick curves such that $\operatorname{mod}_Q\Gamma>0$.

Finally, we define an ordering $\leq$ on curves in Γ. Given $\gamma\in\Gamma$, the distinguished component γ_d cuts V_0 into at least two components. We let $U_+(\gamma)$ denote the component of $V_0\setminus\gamma_d$ the boundary of which (when seen in the chart $V\to[-2,2]\times[0,1]$) contains the arc $[-1,1]\times\{1\}$, and $U_-(\gamma)$ the one whose boundary, when similarly viewed, contains $[-1,1]\times\{0\}$; these boundary arcs are indicated in bold in the figure.

The fact that thick curves do not cross implies that given $\gamma,\gamma'\in\Gamma$, either $U_+(\gamma)\subset U_+(\gamma')$ and $U_-(\gamma)\supset U_-(\gamma')$, or $U_+(\gamma)\supset U_+(\gamma')$ and $U_-(\gamma)\subset U_-(\gamma')$. In the former case, we write $\gamma\geq\gamma'$.

6.1.3 Definition of the tangent space

The tangent space will be a based at a density point y of a set A which we now define. We may assume each $\gamma\in\Gamma$ is parametrized by arc length, the parameter lying in a compact interval I_γ, and is extended so as to be completely parametrized. We may assume Γ contains all complete parametrizations of its curves. For $\gamma\in\Gamma$ and $t\in I_\gamma$ we let

$$f_k(\gamma,t)=\sup_{|h|\leq 1/k}\left(1-\frac{d(\gamma(t-h),\gamma(t+h))}{2|h|}\right);$$

this measures how much γ deviates from being geodesic near t. Since γ is rectifiable and parametrized by arclength, Proposition 4.1 ensures that $f_k(\gamma, t) \to 0$ as $k \to \infty$ for all $\gamma \in \Gamma$ and almost all $t \in I_\gamma$. Set, for $x \in \operatorname{supp}\Gamma \cap V$,

$$g_k(x) = \min\{f_k(\gamma, t) : \gamma \in \Gamma, \gamma(t) = x\}$$

and let A be the set of points x such that $\lim_k g_k(x) = 0$. We claim A has positive measure. Otherwise, we could define an admissible metric ρ by choosing $\rho = \infty$ on A^*, and $\rho = 0$ otherwise. This function is L^Q integrable, and for all $\gamma \in \Gamma$ and almost all $t \in I_\gamma$, $\{f_k(\gamma, t)\}_k$ tends to zero, implying that $\int_\gamma \rho = \infty$ for all $\gamma \in \Gamma$: this would contradict $\operatorname{mod}_Q \Gamma > 0$. Therefore, A has positive measure. By Egorov's theorem, we may assume that (g_k) tends to zero uniformly on A. We pick a point of t-density $y \in A$, which exists by Lemma 3.10.

The point y belongs to the image of a curve γ. Fix a positive sequence (r_n) tending to 0. Without loss of generality, we may assume that $(X_n, d_n, y) = (X, d/r_n, y)$ tends to a metric plane (T, t), and that $(f^{k_n})_n$ tends to a map $h : T \to X$ with a discrete branch set (Theorem 3.2 and Proposition 3.12).

The definition of Γ, the definition of A, and Proposition 4.2 imply that every point in T belongs to the image of a geodesic $\gamma : \mathbb{R} \to T$ which is a limit of a sequence (γ_k) of rescaled thick curves passing through points of A. We denote by $\mathcal{F}_T$ the set of geodesics γ obtained in this way. Note that $\mathcal{F}_T$ is closed with respect to the compact-open topology.

Lemma 6.8 *With the notation from above, the geodesic γ separates T into two simply connected regions $U_+(\gamma)$ and $U_-(\gamma)$, each of which is the interior of the respective limit of $\overline{U_+(\gamma_k)}$ and $\overline{U_-(\gamma_k)}$.*

Proof: The curve γ is a simple curve, unbounded on both sides: considering the Alexandroff compactification of T, $\overline{\gamma}$ becomes a simple closed curve so that the Jordan theorem implies that $T \setminus \gamma$ is the union of two disjoint open disks.

Denote by $K_\pm(\gamma)$ the set of points $w \in T$ arising as limits (as $k \to \infty$) of sequences of points $w_k \in U_\pm(\gamma_k)$, respectively. Let $w \in K_+(\gamma) \cap K_-(\gamma)$, and (w_k), (w'_k) be sequences as above which tend to w; by the bounded turning (BT) property, we may find continua $C_k \subset X_{n_k}$ joining w_k and w'_k with $\operatorname{diam} C_k \lesssim |w_k - w'_k|$. By the Jordan curve theorem, it follows that $C_k \cap \gamma_k \neq \emptyset$. As k tends to infinity, we obtain that $w \in \gamma$. Hence $K_+(\gamma) \cap K_-(\gamma) \subset \gamma$.

Let us prove that $K_+(\gamma) \cup K_-(\gamma) = T$. Pick $w \in T$, and $(w_k) \in \prod X_{n_k}$ which tends to w. Either, there are infinitely many k such that $w_k \in \overline{U_+(\gamma_k) \cup U_-(\gamma_k)}$, in which case $w \in K_+(\gamma) \cup K_-(\gamma)$. Or, w_k is contained in a bounded component W of $X_{n_k} \setminus \gamma_k$ for all k large enough.

In the latter case, we apply Proposition 6.4 to obtain $s < t$ such that $\gamma_k(s) = \gamma_k(t)$, $\gamma_k[s, t]$ separates w_k from $U_+(\gamma_k) \cup U_-(\gamma_k)$ and $\gamma_k(s) \in \partial(U_+(\gamma_k) \cup U_-(\gamma_k))$.

Since $\gamma_k(s) = \gamma_k(t)$, we have

$$|s - t| = |\gamma(s) - \gamma(t)| \leq 2\|\gamma_k - \gamma\|_\infty$$

so that $\operatorname{diam}\gamma_k([s,t]) \le 2\|\gamma_k - \gamma\|_\infty$. Property (ALC2) now implies that

$$\mathrm{diam} W \le 2L \mathrm{diam}\gamma_k([s,t]) \le 4L\|\gamma_k - \gamma\|_\infty \,.$$

It follows that we may find $w'_k \in \overline{U_+(\gamma_k) \cup U_-(\gamma_k)}$ with

$$|w_k - w'_k| \lesssim \|\gamma - \gamma_k\|_\infty \,.$$

This proves that $w \in K_+(\gamma) \cup K_-(\gamma)$ so $K_+(\gamma) \cup K_-(\gamma) = T$. The proof follows.

■

Remark 6.9 *If $\gamma, \gamma' \in \mathcal{F}_T$, then by Corollary 5.3 they cannot cross at a regular point of branched covering $h : T \to X$.*

6.1.4 Simultaneous zooms of thick curves

In this section and the next, we examine the structure of the set of geodesics $\mathcal{F}_T$ in the metric plane T constructed in the previous subsection. Recall that the geodesics in $\mathcal{F}_T$ were obtained as limits of rescaled curves in the family Γ defined in § 6.1.2.

Let $\gamma, \gamma' \in \mathcal{F}_T$. We shall write $\gamma \le \gamma'$ if there exist a subsequence $(n_k)_k$ and sequences of curves (γ_k) and (γ'_k) such that $\gamma_k, \gamma'_k \subset X_{n_k}$, $\gamma_k \le \gamma'_k$ and if both sequences converge to γ and γ' respectively.

We now fix $\gamma, \gamma' \in \mathcal{F}_T$ and assume that there exist a subsequence $(n_k)_k$ and sequences of curves (γ_k) and (γ'_k) with $\gamma_k, \gamma'_k \subset X_{n_k}$ and such that both sequences converge to γ and γ' respectively. For each k, either $\gamma_k \le \gamma'_k$ or $\gamma'_k \le \gamma_k$. So, extracting a subsequence if necessary, we may assume that $\gamma \le \gamma'$. We may also assume that $\overline{U_\pm(\gamma_k)}$ and $\overline{U_\pm(\gamma'_k)}$ converge to closed and connected domains $K_\pm(\gamma)$ and $K_\pm(\gamma')$, cf. Lemma 6.8.

Lemma 6.10 *If $\gamma \ne \gamma'$, then, for any $z \in U_+(\gamma) \cap U_-(\gamma')$, there exists a curve $\gamma_z \in \mathcal{F}_T$ going through z such that $\gamma \le \gamma_z \le \gamma'$.*

Proof: Suppose $z_n \to z$, $z_n \in X_n$. We may assume that $z_n \in U_+(\gamma_n) \cap U_-(\gamma'_n)$. By Lemma 3.11, there is a sequence of curves $(\gamma_{z,n})_n$ such that $d_n(z_n, \gamma_{z,n})$ tends to 0. It follows that $\gamma_n \le \gamma_{z,n} \le \gamma'_n$. Extracting a limit, the lemma follows.

■

Lemma 6.11 *If γ and γ' intersect, then $\gamma = \gamma'$.*

Proof: We assume that γ and γ' are distinct but that $\gamma(0) = \gamma'(0) = z_0$ and $\gamma((-a,0)) \cap \gamma' = \gamma'((-a,0)) \cap \gamma = \emptyset$.

Let $z \in U_+(\gamma) \cap U_-(\gamma')$. By Lemma 6.10, there exists a curve $\gamma_z \in \mathcal{F}_T$ going through z such that $\gamma \le \gamma_z \le \gamma'$ For any n, there exists a continuum C_n joining $\gamma_n(0)$ to $\gamma_n'(0)$ with $\operatorname{diam} C_n \lesssim |\gamma_n(0) - \gamma_n'(0)|$. By the Jordan curve theorem, $C_n \cap \gamma_{z,n} \neq \emptyset$, so that $z_0 \in \gamma_z$.

We note that z_0 belongs to three limits of thick curves. If h is a local homeomorphism at z_0, then we obtain a contradiction by Proposition 6.5.

We assume now that h is not a local homeomorphism at z_0. Set $w_0 = h(z_0)$. By Corollary 3.14, we may assume that in local coordinates near z_0 and w_0, the map h is given by $h : \mathbb{D} \to \mathbb{D}$, $h(z) = z^k$, for some $k \ge 2$; in particular, this restriction is proper. For each element $c \in \mathcal{F}_T$ which contains the origin, we may restrict c and reparametrize it so that $c : [-1, 1] \to \overline{\mathbb{D}}$ is a crosscut with $c(0) = 0$.

We now establish some facts about crosscuts c obtained as the restrictions of elements C of $\mathcal{F}_T$ with $\gamma \le C \le \gamma'$.

1. Since h is proper on $\mathbb{D}$, $h(c)$ joins 0 to the boundary of $\mathbb{D}$.

2. Corollary 5.3 implies $h(c)$ is also thick, so $h(c) \cap h(\gamma) = \{0\}$ since both images are thick curves.

3. The map h is locally injective on $\mathbb{D} \setminus \{0\}$ and $h^{-1}(0) = 0$, so, if $(h \circ c)|_{[-1,0]}$ was not injective, then there would be different times $s < t$ in $(-1, 0)$ such that $h(c(s)) = h(c(t))$. But then the thick curves $c|_{[-1,(2s+t)/3]}$ and $c|_{[(s+2t)/3,0]}$ intersect at a regular point of h, which is impossible. Similarly, $(h \circ c)|_{[0,1]}$ is also injective. It also follows that if $h \circ c$ is not globally injective, then $c([-1, 0]) = c([0, 1])$.

4. Furthermore, there may be at most $(k - 1)$ ordered curves which are mapped 2-to-1 under $h|_{\mathbb{D}}$, so that we may as well consider that $h(c \cap \mathbb{D})$ is an arc for every restriction of a curve $c \in \mathcal{F}_T$ which defines a crosscut and which goes through the origin such that $\gamma \le c \le \gamma'$.

Given the curve γ in the lemma, we let Ω be the connected component of $\mathbb{D} \setminus h^{-1}(h(\gamma([-1, 0])))$ which contains γ in its boundary and intersects $U_+(\gamma)$. Note that $h|_\Omega : \Omega \to \mathbb{D} \setminus h(\gamma((-1, 0]))$ is a homeomorphism. For any other curve $c \in \mathcal{F}_T$, we know that if $h(c)$ is not a subset of $h(\gamma)$, then c cannot intersect $h^{-1}(h(\gamma([-1, 0))))$; therefore, we may define $\widehat{c} \subset \Omega \cup \gamma([-1, 0])$ as the lift of $h(c)$ to $\Omega \cup \gamma([-1, 0])$.

It follows from above that $\widehat{\gamma'} \cap \Omega$ is contained in a component of $\Omega \setminus \widehat{\gamma}$, for otherwise $h(\gamma')$ and $h(\gamma)$ would cross. Let V' be the connected component of $\Omega \setminus \widehat{\gamma'}$ with boundary $\widehat{\gamma'}$. We first consider the case that V' is not a subset of $U_+(\gamma')$. Then, for any $z \in V' \setminus U_+(\gamma')$, Lemma 6.10 provides us with a curve $\gamma_z \in \mathcal{F}_T$ such that $\gamma \le \gamma_z \le \gamma'$. In particular $0 \in \gamma_z$. Since $\gamma_z \cap V' \neq \emptyset$, it follows that $\widehat{\gamma_z} \subset V' \cup \{0\}$ so that $h(\gamma)$, $h(\gamma')$ and $h(\gamma_z)$ satisfy the assumptions of Proposition 6.5, yielding a contradiction. Therefore, $V' \subset U_+(\gamma')$. We consider a new curve γ'' given by Lemma 6.10 such that $\gamma \le \gamma'' \le \gamma'$ and $\gamma'' \cap \mathbb{D}$ is not contained in $U_+(\gamma')$: let V'' be the connected component of $\Omega \setminus \widehat{\gamma''}$ with boundary $\widehat{\gamma''}$. By construction, V'' is not a subset of $U_+(\gamma')$. As above, we consider a curve $\gamma_z \in \mathcal{F}_T$ such that $\gamma \leq \gamma_z \leq \gamma'$ and $\widehat{\gamma_z} \subset V'' \cup \{0\}$: we obtain another contradiction from $h(\gamma)$, $h(\gamma_z)$ and $h(\gamma'')$.

■

6.1.5 Different zooms

Let us assume that $\gamma, \gamma' \in \mathcal{F}_T$ are two distinct curves.

This implies that there are two increasing functions $\varphi, \psi : \mathbb{N} \to \mathbb{N}$, and thick curves $\gamma_{\varphi(n)} \subset X_{\varphi(n)}$ and $\gamma'_{\psi(n)} \subset X_{\psi(n)}$ which tend to γ and γ' respectively.

Lemma 6.12 *The curves γ and γ' are disjoint.*

Proof: By the previous section, we may assume that the sequences φ and ψ have no integers in common. We will show that if γ and γ' are not disjoint, then we can build a curve γ'' which will cross γ' at a regular point of h, contradicting Corollary 5.3.

We may assume that $\gamma(0) = \gamma'(0)$. Since h has a discrete branch set (Corollary 3.14), we may assume that $\gamma'|_{(0,a)}$ has no branch points of h, and that it is disjoint from γ. Let $z_1 = \gamma'(a/2)$, and apply Lemma 6.10 to obtain a curve γ_1 going through z_1 obtained by blowing up thick curves from $X_{\varphi(n)}$; we may assume that $\gamma \leq \gamma_1$. If γ_1 does not cross γ', then we construct a similar curve $\gamma_2\,(\neq \gamma_1)$ intersecting γ' such that $\gamma \leq \gamma_2 \leq \gamma_1$. It follows that γ_2 has to cross γ', since it is disjoint from γ and γ_1 by Lemma 6.11.

■

6.1.6 Foliations

We have proved that any point in T belonged to a unique element of $\mathcal{F}_T$, and that no two such curves could intersect.

It remains to prove that $\mathcal{F}_T$ is a genuine foliation. Following Whitney [Whi, Part II], one has to check that this is a *regular family of curves*: given any point p and a direction on the curve through p, there is an arc pq in this direction with the property that for every $\epsilon > 0$ there is a $\delta > 0$ such that, for any point p' within a distance δ of p, there is an arc $p'q'$ of the curve through p' which lies within an ϵ-neighborhood of pq and on which q' lies within an ϵ-neighborhood of q; moreover, if r' and s' are two points on $p'q'$ within a distance δ of each other, then the diameter of the arc $r's'$ is less than ϵ (see also [Kol, Lme 4.7] for an explicit construction of a transversal arc).

The second condition is automatically satisfied since curves are geodesics of T. For the first, assume that it does not hold: then, given an arc from $\mathcal{F}_T$ joining two points p and q, there is some $\epsilon_0 > 0$ with the property that there is a sequence of points (p_n) tending to p such that no subarc $(p_n q_n)$ either leaves the ϵ_0-neighborhood of (pq) or q_n is ϵ_0-away from q. If we consider parametrizations of the curves γ_n going through p_n with $\gamma_n(0) = p_n$, then, Ascoli's theorem implies that the sequence (γ_n) tends uniformly on compact sets to a geodesic γ going through p; since $\mathcal{F}_T$ is closed, it follows that $\gamma \in \mathcal{F}_T$, contradicting the definition of (p_n). Therefore $\mathcal{F}_T$ is a regular family of curves, hence a genuine foliation.

This ends the proof of Proposition 6.2.

Before proving Corollary 6.3, we note the following:

Lemma 6.13 *Suppose $\mathcal{F}_1, \mathcal{F}_2$ are two nonsingular foliations of an open subset U of X, each of whose leaves are locally thick curves. Then $\mathcal{F}_1 = \mathcal{F}_2$.*

Proof: Let $x \in U$. The leaves L_1, L_2 in $\mathcal{F}_1, \mathcal{F}_2$ containing x cannot cross at x since they are both thick. If they are not the same, there is a leaf L_1' near L_1 which crosses L_2, which is impossible.

∎

Proof: (Cor. 6.3) We push the foliation $\mathcal{F}_T$ down to a singular foliation $\mathcal{F}_X$ of X using h: away from the images of branch points of h, we define the leaves of $\mathcal{F}_X$ to be the images of the leaves of $\mathcal{F}_T$ under h by Corollary 5.3, they are locally thick curves, so Lemma 6.13 implies $\mathcal{F}_X$ is well-defined.

The construction of the tangent T implies that there exists a compact set $K \subset T$ such that $h(int(K)) = X$. Since h is a branched covering, its branch locus B_h is discrete, hence its intersection with K is finite. For any regular point z of h in $int(K)$, $h_*\mathcal{F}_T = \mathcal{F}_X$ is regular in the neighborhood of its image $h(z)$. It follows that the set F of singular points of $\mathcal{F}_X$ is contained in the image $h(int(K) \cap B_h)$ and is therefore finite.

We now analyze the structure of the singularities of $\mathcal{F}_X$. Let x_0 be a branch point of h; we use the fact that T and X are surfaces and h is an open map: by Corollary 3.14, there are neighborhoods U and V of x_0 and $h(x_0)$ respectively such that $h : (U, x_0) \to (V, h(x_0))$ is equivalent to $z \mapsto z^k$ in the unit disk of the complex plane $\mathbb{C}$ for some positive integer $k \geq 2$. Let γ be the connected component in U of the leaf of $\mathcal{F}_T$ which contains x_0; since $h|_U$ is proper onto V, there is some $j \in \{1, 2\}$ such that its image $h(\gamma)$ cuts V into j components, so that $h^{-1}(h(\gamma))$ cuts U into $j \cdot k$ components. Since thick curves on X cannot cross, it follows that $h^{-1}(h(\gamma)) \cap U = \gamma$ and $j \cdot k = 2$: hence $k = 2$ and $j = 1$. Therefore, the branched covering $h : T \to X$ can have only simple critical points and a curve which goes through a critical point of h must map in a locally 2-to-1 fashion onto its image: all singularities of $\mathcal{F}_X$ are prongs.

We now prove the invariance of the foliation $\mathcal{F}_X$ under f: if $x \in X$ is not a branched point of f, then $x \in F$ if and only if $f(x) \in F$. Furthermore, if $f(x)$ is a regular point of $\mathcal{F}_X$, then, pulling back $\mathcal{F}_X$ under f^{-1} to a neighborhood of x, we see that x cannot be a branched point for $\mathcal{F}_X$ would have a singularity which would not be a prong: the invariance of $\mathcal{F}_X$ is established.

This ends the proof of Corollary 6.3.

∎

6.2 Parabolic orbifold structure

In this subsection, we prove that f is conjugate to a Lattès example. We will rely on the "easy part" of Douady and Hubbard's classification of finite branched coverings of the sphere [DH].

Proof: (Theorem 6.1) By Corollary 6.3, we know that $f^{-1}(\mathcal{F}_X \setminus F) \subset \mathcal{F}_X \setminus F$. It follows that $f^{-1}(F) \subset (B_f \cup F)$, so that $P_f \subset F$ and f is postcritically finite. Note that if f is locally injective at x, and if $y = f(x) \in F$, then $x \in F$. This implies that in fact $P_f = F$, that $f^{-1}(P_f) = P_f \cup B_f$. ¿From [DH, Lma 3.2] it follows that $\#P_f \leq 4$ and more precisely that in fact $\#P_f = 4$. Since all the singularities of $\mathcal{F}_X$ are prongs, the branch points of f and of h are simple, so the orbifold associated to f is the $(2,2,2,2)$-orbifold. We may now apply [DH, Prop. 9.3] to deduce that f lifts as a covering map g of a torus $\tilde{X}$. Since f is topologically cxc, g is positively expansive. Pick a basis for $H_1(\tilde{X}, \mathbb{Z})$ and let A be the matrix of the induced map $H_1(g) : H_1(\tilde{X}, \mathbb{Z}) \to H_1(\tilde{X}, \mathbb{Z})$. Since g is positively expansive, g and the map on the torus $\mathbb{R}^2/\mathbb{Z}^2$ induced by A are topologically conjugate. Hence f is topologically conjugate to the Lattès example f_A induced by A.

■

6.3 Tangents as universal orbifold covering

In this subsection, we establish the claim in Remark 1.9.

We begin with an easy consequence of the analysis in the preceding subsection:

Corollary 6.14 *Under the hypothesis of Theorem 6.1, the tangent map $h : T \to X$ constructed in § 6.1.3 is the universal orbifold covering map of the orbifold associated to f.*

Proof: Since thick curves cannot cross, every branch point of h is simple and maps to an element of F and, conversely, any preimage of a point in F under h is a simple branch point of h.

■

The construction of the above tangent space T is very indirect. We continue with a proposition which gives a direct construction in which the basepoint is a periodic point.

Proposition 6.15 *Let $f : S^2 \to S^2$ be a topological cxc map, p be a periodic point of f of period k, and suppose there is a neighborhood W of p such that $f^k : W \to f^k(W)$ is a homeomorphism with $\overline{W} \subset f^k(W)$. Let $X \in \mathcal{G}(f)$.*

Then there exists a tangent space (T, t) to X at p locally homeomorphic to W, an expanding homeomorphism $\psi : (T, t) \to (T, t)$ fixing t whose iterates are uniformly quasisymmetric, and a quasiregular map $h : (T, t) \to (X, p)$ such that $h \circ \psi = f^k \circ h$, $h(t) = p$.

In the case of the sphere equipped with a visual metric, i.e. $X = (S^2, d_v)$, the map h is an isometry near t, and the iterates of ψ are similarities with constant expansion factor.

Here, by quasiregular, we mean an open map which is locally uniformly quasisymmetric away from its discrete branch set.

Proof: By Lemma 3.13, it is enough to prove the proposition with $d = d_v$.

Let $g : (W, p) \to (f^{-k}(W), p)$ be the local inverse of f^k near p. We may assume that $|g(x) - g(y)| = \lambda |x - y|$ with $\lambda = \theta^k$, for all $x, y \in W$ by [HP1, Prop. 3.2.3]; here, θ is the constant given by Theorem 1.5.

Define scaling functions $\sigma_n : (W, d) \to W_n = (W, d/\lambda^n)$ and let $g_n : W_n \to W_{n+1}$ be defined as $g_n = \sigma_{n+1} \circ g \circ \sigma_n^{-1}$, which is an isometric embedding.

We may then consider the following inductive limit:

$$\mathcal{W} = \varinjlim (W_n, g_n) .$$

Since the g_n are isometries, the set $\mathcal{W}$ is naturally a metric space. For all n, W_n embeds canonically in $\mathcal{W}$. Define $\psi : \mathcal{W} \to \mathcal{W}$ by $\psi_n(x) = \sigma_{n+1} \circ \sigma_n^{-1}(x)$, for $x \in W_n$. We check that

$$\begin{aligned} \psi_{n+1} \circ g_n &= \sigma_{n+2} \circ \sigma_{n+1}^{-1} \circ \sigma_{n+1} \circ g \circ \sigma_n^{-1} \\ &= \sigma_{n+2} \circ g \circ \sigma_{n+1}^{-1} \circ \sigma_{n+1} \circ \sigma_n^{-1} \\ &= g_{n+1} \circ \psi_n . \end{aligned}$$

It follows that

$$|\psi(x) - \psi(y)| = \frac{1}{\lambda} |x - y| .$$

If $x \in W_n$, let $h_n(x) = f^{kn} \circ \sigma_n^{-1}(x)$. This defines a 1-Lipschitz map $h : \mathcal{W} \to X$ since

$$\begin{aligned} h_{n+1} \circ g_n &= f^{kn} \circ f^k \circ \sigma_{n+1}^{-1} \circ \sigma_{n+1} \circ g \circ \sigma_n^{-1} \\ &= f^{kn} \circ (f^k \circ g) \circ \sigma_n^{-1} = h_n . \end{aligned}$$

One has

$$\begin{aligned} h_{n+1} \circ \psi_n &= f^{kn+k} \circ \sigma_{n+1}^{-1} \circ \sigma_{n+1} \circ \sigma_n^{-1} \\ &= f^k \circ (f^{kn} \circ \sigma_n^{-1}) = f^k \circ h_n(x) . \end{aligned}$$

Therefore, $h \circ \psi = f^k \circ h$.

Since the maps

$$\sigma_n : \left(W, \frac{d}{\lambda^n}, p \right) \to (W_n, p) \subset \mathcal{W}$$

are isometries, it follows that $\mathcal{W}$ is the Gromov-Hausdorff limit of $(X, d_v/\lambda^n, p)$. Note that if $W = W_0$ is chosen sufficiently small, then $h|_{W_1}$ is an isometry, by construction.

■

Remark 6.16 *If $d = d_v$, one of the visual metrics given by Theorem 1.5, then the assumption [Deg] in Proposition 6.15 can be omitted, provided that the branch set is disjoint from the cycle containing p.*

We conclude this section with the proof of the claim in Remark 1.9. The Lefschetz formula shows that f has a fixed-point, p. Let (T, t) be the tangent space at p, $h : (T, t) \to (X, p)$ be the quasiregular map, and $\psi : (T, t) \to (T, t)$ be the associated expanding homothety, each given by Proposition 6.15. Let $\mathcal{F}_X$ be the foliation of X by thick curves given by Corollary 6.3.

Restricting the neighborhood W if necessary, we may assume that $\mathcal{F}_X \cap W$ has at most one singularity —which would be at p— and that there is a neighborhood $N \subset T$ of t such that $h : N \to W$ is quasisymmetric. The foliation $\mathcal{F}_N = h^{-1}(\mathcal{F}_X) \cap N$ has at most one singularity —at t— and all the leaves of $\mathcal{F}_N$ are thick by Proposition 4.6. Applying iterates of ψ, we obtain a ψ-invariant foliation $\mathcal{F}_T$ of T by locally thick curves with at most one singularity such that $h : (T, \mathcal{F}_T) \to (X, \mathcal{F}_X)$, since $\cup_n \psi^n(N) = T$.

If $\mathcal{F}_T$ has no singularity at t, then the same analysis as in Corollary 6.14 shows that $h : T \to X$ is again a universal orbifold cover. In the notation of Remark 1.9, we set $\tilde{X} = T$, $\pi = h$, and $\psi = \psi$. If t is a singularity of $\mathcal{F}_T$, let $q : (\tilde{T}, \tilde{t}) \to (T, t)$ be a double-cover of T ramified above t (which exists, since T is a plane). The metric on T lifts to $\tilde{T}$ so that q becomes a local isometry away from $\tilde{t}$. Then $h \circ q : \tilde{T} \to X$ is now the universal orbifold cover, and ψ lifts to $\tilde{\psi} : \tilde{T} \to \tilde{T}$. We set $\tilde{X} = \tilde{T}$, $\pi = h \circ q$, and $\psi = \tilde{\psi}$.

∎

7 Lattès or rational

This section is devoted to the proof of Theorems 1.1 and 1.2 in the case of iterated maps. We let $f : S^2 \to S^2$ be topologically cxc and we assume that there is an Ahlfors-regular metric space $X \in \mathcal{G}(f)$ of dimension $Q = \mathrm{confdim}_{AR}(f)$. ¿From Propositions 5.1 and 4.4, we obtain a family Γ of thick curves of positive Q-modulus on X.

If there are two thick curves which cross, then Proposition 4.17 implies that f is conjugate to a rational map. By [HP1, Cor. 4.4.2], this rational map is semihyperbolic.

If thick curves do not cross, then Corollary 6.3 implies the existence of an invariant foliation —completing the proof of Theorem 1.2— and Theorem 6.1 implies that f is conjugate to a Lattès map f_A.

Since f_A is expanding, the eigenvalues of A have modulus larger than one. Since the conformal gauge is an invariant of topological conjugacy [HP1, § 2.8], the conformal dimensions of f and of f_A are the same. We cannot have $Q = 2$, otherwise there would be crossing thick curves. Therefore, $Q > 2$ and Theorem 1.7 implies that A has two distinct real eigenvalues.

∎

A Further applications of the comparison formula

Here, we sketch some applications of Proposition 4.10 and of its corollaries. For brevity, we refer the reader to the cited references for the relevant definitions and background.

A.1 The Loewner property and its combinatorial version

Following J. Heinonen and P. Koskela [HK], we say that an arcwise connected metric space is Q-Loewner, $Q > 1$, if there is a non-increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathrm{mod}_Q(E,F) \geq \psi(\Delta(E,F))\,,$$

for any pair of disjoint continua E and F in X and with

$$\Delta(E,F) = \frac{\mathrm{dist}(E,F)}{\min\{\mathrm{diam}E, \mathrm{diam}F\}}\,.$$

In [Kle], B. Kleiner suggests a combinatorial version of a Q-Ahlfors regular and Q-Loewner space. Let X be a proper metric space. Let $\mathcal{R}_n$ denote a maximal 2^{-n}-separated set of X, then $\mathcal{S}_n = \{B(x,1/2^n), x \in \mathcal{R}_n\}$ defines a uniform sequence of quasipackings.

Say X satisfies the *combinatorial Q-Loewner property* if there are non-increasing positive functions ψ and φ such that

$$(CLP) \qquad \psi(\Delta(E,F)) \leq \mathrm{mod}_Q(E,F,\mathcal{S}_n) \leq \varphi(\Delta(E,F))\,,$$

for any pair of disjoint continua E and F in X and for any n large enough (with respect to (E,F)).

The following notion also appears in [Kle]: a compact metric space X is *selfsimilar* if there is a constant $L_0 \geq 1$ such that for any ball $B(x,r) \subset X$ with $r \in (0, \mathrm{diam}\, X]$, there is an open set $U \subset X$ which is L_0-bi-Lipschitz to the rescaled ball $(B(x,r),(1/r)d)$.

In [BdK], M. Bourdon and B. Kleiner prove that, for a selfsimilar space X, for $\delta > 0$ small enough, and for any $p \geq 1$,

1. if $\{\mathrm{mod}_p(\Gamma_\delta, \mathcal{S}_n)\}_n$ is bounded, then there exists a decreasing function ϕ such that the upper bound of (CLP) holds;

2. the sequence $\{\mathrm{mod}_p(\Gamma_\delta, \mathcal{S}_n)\}_n$ is essentially submultiplicative;

3. there is a critical dimension $Q_M > 0$ such that $\{\mathrm{mod}_p(\Gamma_\delta, \mathcal{S}_n)\}_n$ tends to 0 for $p > Q_M$, is unbounded for $1 \leq p < Q_M$ and admits a positive lower bound for $p \in [1, Q_M]$.

The question arises whether the dependence on p behaves like the Hausdorff dimension. If we assume that there exists $Q > 1$ such that

$$\mathrm{mod}_Q(\Gamma_\delta, \mathcal{S}_n) \asymp 1\,,$$

e.g. if X satisfies the combinatorial Loewner property, then $p \mapsto \{\mathrm{mod}_p(\Gamma_\delta, \mathcal{S}_n)\}_n$ has such a behavior according to Corollary 4.11: the moduli tend to infinity for $p < Q$ and to zero for $p > Q$ (and $Q = Q_M$).

Let X be the boundary of a hyperbolic Coxeter group endowed with a visual metric. For positive δ and r small enough, M. Bourdon and B. Kleiner define the family $\mathcal{F}^g \subset \Gamma_\delta$ of so-called generic curves i.e., this is the subfamily of Γ_δ such that none of these curves is contained in the r-neighborhood of a sub-Coxeter, also called a parabolic, subgroup. They introduce the critical exponent

$$Q_m = \sup\{p \geq 1,\ \lim \mathrm{mod}_p(\mathcal{F}^g, \mathcal{S}_n) = +\infty\}\,.$$

If X satisfies the combinatorial Loewner property, then $Q_m = Q_M$: this follows from choosing two disjoint continua $\{E, F\}$ such that the family of curves joining them is contained in $\mathcal{F}^g$. This provides an answer to a question which was raised by M. Bourdon and B. Kleiner; see [BdK, Rmk (1), § 3].

A.2 Combinatorial moduli on surfaces

The proof of Theorem 1.6 relies heavily on [BnK1, Thm. 1.1]:

Theorem A.1 (M. Bonk & B. Kleiner) *Suppose X is a metric space which is homeomorphic to $\mathbb{S}^2$, linearly locally connected, and Ahlfors 2-regular. Then X is quasisymmetrically equivalent to $\mathbb{S}^2$.*

Note that if X is a surface which satisfies the combinatorial Loewner property in some dimension $Q \geq 2$, then it is fairly easy to construct two families of curves Γ_1 and Γ_2 of positive modulus such that every curve in Γ_1 intersects any curve in Γ_2: it follows at once that $Q = 2$.

In particular, if X is Q-Loewner and Q-regular then $Q = 2$. Hence, if X is furthermore homeomorphic to $\mathbb{S}^2$, then it is also quasisymmetric to $\mathbb{S}^2$: this provides an alternative proof of [BnK1, Thm. 1.2].

A.3 Conformal dimension and Gromov hyperbolic groups with 2-sphere boundary

We now sketch how Theorem 1.3 may be proved using Theorem 1.2.

The arguments in [BnK3] used to prove Theorem 1.3 proceed by first showing that there exists an Ahlfors-regular Loewner metric in the gauge of the group, and then applying the uniformisation theorem for Loewner spheres [BnK1].

Here, we sketch how to bypass the Loewner property and prove instead that the boundary admits a 2-regular metric.

Proof: (Thm 1.3) The first step is to prove that there is a family of positive modulus on the boundary of G: the argument is the same as above. We use [BnK2, Lma 5.3] which says that any weak tangent is quasi-Möbius equivalent, say via a map h, to ∂G, punctured at some point, cf. [BnK3, Cor. 1.6].

Then it is enough to prove that there are two thick curves which cross. For this, we may assume that this is not the case: the argument for the proof of Proposition 6.2 works the same: the same blowup strategy defines a weak tangent space foliated by locally thick geodesic curves. We push this foliation forward via h to ∂G. Since the group acts by quasi-Möbius homeomorphisms, it preserves the set of thick curves, hence the foliation; this in turn implies that G has a global fixed point (the puncture): contradiction. Thus, there are two thick curves which cross, which implies that $Q = 2$. An application of Theorem 1.6 concludes the proof.

■

B Applications to general topologically cxc maps

In [HP1], topologically cxc maps are defined in a much general setting than on spheres. We recall briefly the definition and record the results which hold in this more general context.

Suppose X, Y are locally compact Hausdorff spaces, and let $f : X \to Y$ be a finite-to-one continuous map. The *degree* of f is

$$\deg(f) = \sup\{\# f^{-1}(y) : y \in Y\}.$$

For $x \in X$, the *local degree* of f at x is

$$\deg(f; x) = \inf_U \sup\{\# f^{-1}(\{z\}) \cap U : z \in f(U)\}$$

where U ranges over all neighborhoods of x.

The map $f : X \to Y$ is a *finite branched covering* (abbreviated fbc) provided $\deg(f) < \infty$ and

(i)
$$\sum_{x \in f^{-1}(y)} \deg(f; x) = \deg f$$
holds for each $y \in Y$;

(ii) for every $x_0 \in X$, there are compact neighborhoods U and V of x_0 and $f(x_0)$ respectively such that

$$\sum_{x \in U, f(x)=y} \deg(f;x) = \deg(f;x_0)$$

for all $y \in V$.

Let $\mathfrak{X}_0, \mathfrak{X}_1$ be Hausdorff locally compact, locally connected topological spaces, each with finitely many connected components. We further assume that $\mathfrak{X}_1$ is an open subset of $\mathfrak{X}_0$ and that $\overline{\mathfrak{X}_1}$ is compact in $\mathfrak{X}_0$. The *repellor* of $f : \mathfrak{X}_1 \to \mathfrak{X}_0$ is

$$X = \{x \in \mathfrak{X}_1 | f^n(x) \in \mathfrak{X}_1 \ \forall n > 0\} = \bigcap_n \overline{f^{-n}\mathfrak{X}_0}.$$

Let $\mathcal{U}_0$ be a finite cover of X by open, connected subsets of $\mathfrak{X}_1$ whose intersection with X is nonempty. Let $\mathcal{U}_n$, $n \geq 1$, be the connected components of $f^{-n}(U)$, where U ranges over $\mathcal{U}_0$.

We say $f : (\mathfrak{X}_1, X) \to (\mathfrak{X}_0, X)$ is *topologically coarse expanding conformal with repellor* X provided

1. the restriction $f|X : X \to X$ is also an fbc of degree equal to d;
2. there exists a finite covering $\mathcal{U}_0$ as above, such that the axioms [Exp], [Irred] and [Deg] hold.

Without any changes, Theorem 1.5, Proposition 3.8, Corollary 3.9, Proposition 3.12 and Proposition 6.15 hold in this broader context. We conclude with the following statement whose proof follows from the same arguments as on the sphere:

Theorem B.1 *Let $f : (\mathfrak{X}_1, X) \to (\mathfrak{X}_0, X)$ is a topologically cxc map with repellor X. Assume that X is locally connected and that there is an Ahlfors regular distance on X of minimal dimension Q. Then*

1. *There exists a family of curves in X of positive Q-modulus.*
2. *Thick curves are preserved by f.*
3. *There exists $\delta > 0$ such that every point of X belongs to a thick curve of diameter at least δ.*

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